UNIVERSITY OF ABERDEEN **SESSION 2003–04**

Degree Examination

MX3502 Groups and Geometry

Monday 24 May 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal marks.

1. **(a)** Let G be a group with neutral element e and let $x \in G$. Define what is meant by the order $o(x)$ of x.

Define what is meant by $\langle x \rangle$.

Show that if $o(x) = k < \infty$, then $\langle x \rangle$ has exactly k elements.

Show that if $x^m = e$ for an integer $m > 0$, then $o(x)$ divides m.

Show that if $o(x^5) = 3$, then $o(x)$ equals 3 or 15.

Show that if $o(x^3) = 3$, then $o(x) = 9$.

(b) Regard $\mathbb{R}$ as a group using ordinary addition of real numbers. Describe the smallest subgroup H of $\mathbb{R}$ which contains the numbers $1/3$ and $1/7$. As a group in its own right, is H cyclic ?

(c) Define Euler's function $m \mapsto \varphi(m)$, where $m \in \{1, 2, 3, 4, \dots\}$.

By considering a cyclic group of order n, its subgroups and the generators of each subgroup, show that

$$\sum_{m|n} \varphi(m) = n$$

where the sum ranges over the positive integers m dividing n [including 1 and n itself]. Carefully state any general facts about cyclic groups that you are using.

Let H be a finite group with neutral element e. Let m be a positive integer which divides $|H|$. Show that if H has at most m elements x for which $x^m = e$, then either H has $\varphi(m)$ elements of order m or no elements of order m.

Hence show that the multiplicative group $\mathbb{F}^*$ of nonzero elements in a finite field $\mathbb{F}$ is cyclic.

2. **(a)** Let G be a group. Define what is meant by a subgroup of G.

Let H be a subgroup of G. Define what is meant by a left coset of H in G, and by a right coset of H in G. Assuming that G is finite, prove that $|H|$ divides $|G|$ (Lagrange's theorem).

(b) Let G be a group and let $v, w \in G$ be elements such that $wv = v^{-1}w$ and $o(v) = 8$. Let $H = \langle v \rangle$ be the subgroup generated by v.

Suppose that $w^2 \in H$. Show that $H \cup Hw$ is a subgroup of G which has 16 elements. Show also that $w^2 = e$ or $w^2 = v^4$.

(c) Define what is meant by the symmetric group S_n. Define what is meant by an orbit of an element $\sigma \in S_n$. Define what is meant by the sign of σ.

(d) Let $\sigma \in S_n$ and $j \in \mathbb{Z}$. Show that every orbit of σ^j is contained in an orbit of σ.

(e) Let $\lambda \in S_{11}$ be given in Cauchy notation by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 2 & 1 & 4 & 6 & 11 & 9 & 10 & 3 & 7 & 5 \end{pmatrix}.$$

Write λ in cycle notation. Determine $o(\lambda)$, the order of λ. Calculate λ^{15}, also in cycle notation. Determine the orbits of λ and the sign of λ.

3. **(a)** Let F and G be groups. Define what is meant by a *homomorphism* from F to G. Define what is meant by the *kernel* and the *image* of such a homomorphism and what is meant by a *normal* subgroup of F.

Prove that the kernel of a homomorphism from F to G is a normal subgroup of F.

(b) Let G be a group, H a normal subgroup of G. Explain how the set of left cosets G/H can be made into a group. State the "first isomorphism theorem".

(c) Let G be the set of upper triangular real 3×3 matrices with nonzero determinant:

$$G = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{21} & a_{22} \\ 0 & 0 & a_{33} \end{bmatrix} \;\middle|\; a_{ij} \in \mathbb{R},\ a_{11}a_{22}a_{33} \neq 0 \right\}.$$

Ordinary matrix multiplication makes this into a group. [You are not required to verify this.] Let H be the subset of G consisting of all upper triangular 3×3 matrices of the form

$$\begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{22} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

Show that H is a normal subgroup of G and that G/H is isomorphic to the group of upper triangular real 2×2 matrices.

(d) Let G be a cyclic group of order 5. Determine the number of homomorphisms from G to S_5.

4. **(a)** Define what is meant by an action of a group G on a set X. Define what is meant by an orbit of the action. Define what is meant by the stabiliser subgroup $\mathrm{St}(y)$ for an element $y \in X$.

(b) Let H be the set of all 2×2–matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are real numbers satisfying $a + b = c + d$ and $ad - bc \neq 0$. Ordinary matrix multiplication makes this into a group. [You are not required to verify this.] There is a left action of H on $\mathbb{R}^2$ defined in the standard manner by multiplying matrices with column vectors. [You are not required to verify that the conditions for an action are satisfied.] Determine the orbits of this action. Choose an element in each orbit and determine its stabiliser subgroup.

(c) Let G be a finite group, H a subgroup of G. Define what is meant by the index $[G : H]$.

Let G be a finite group and H a subgroup of G such that $[G : H] = n > 1$. By considering the standard action of G on the set of left cosets G/H, prove that there exists a normal subgroup K of G such that $[G : K]$ divides $n!$ and $K \subset H$.

Hence show that there is no simple group of order 130, carefully stating any general theorem that you use.