UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination
MX3020 Group Theory
Friday 19 January 2007 (3 pm to 5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define what is meant by a subgroup of a group G. In each of the following, decide whether H is a subgroup of the group G.

(i) $G = GL(2, \mathbb{R})$,

$$H = \{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \ : \ a, b \in \mathbb{R},\ a \neq 0\}$$

(ii) $G = S_5$, H is the subset consisting of all elements σ of G such that $\sigma^3 = (1)$.

(b) Let G be a group and H a subgroup of G. Define what is meant by a left coset of H in G. State Lagrange's theorem.

(c) Let G be a group such that $g^2 = 1$ for all $g \in G$. Show that G is abelian.

2. **(a)** Let G and H be groups.

Define what is meant by a homomorphism from G to H. Define what is meant by the kernel of a homomorphism from G to H. Define what is meant for a subgroup N of G to be a normal subgroup of G.

Show that the kernel of a homomorphism from G to H is a subgroup of G and that it is a normal subgroup of G.

(b) Let $\mathbb{R} - \{0\}$ be the group of non-zero real numbers under multiplication. Show that the groups $GL(2, \mathbb{R})/SL(2, \mathbb{R})$ and $\mathbb{R} - \{0\}$ are isomorphic. Clearly state any theorem which you use to prove this.

(c) Let G be a group and let $g \in G$. Define what is meant by the subgroup generated by g. Define what it means for G to be cyclic.

Let p be a prime number. Show that if $|G| = p$, then G is cyclic. Clearly state any theorem you use.

3. **(a)** Let G be a group and let $g \in G$. Define what is meant by the order $o(g)$ of g.

Suppose that $o(g) = 99$. Find $o(g^7)$ and $o(g^9)$. Give a brief explanation for your answer.

(b) Let σ be the following permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ given in two-line notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}$$

(i) Write σ in cycle notation.

(ii) Is σ an element of A_7?

(c) Let τ be an element of S_{14} with cycle type $1, 1, 2, 3, 3, 4$.

(i) What is the order of τ? Give a brief explanation for your answer.

(ii) How many subgroups does $\langle \tau \rangle$ have? What are their orders? Give a brief explanation for your answer.

(d) Write down all the conjugates of (123) in S_4. Give a brief explanation for your answer.

(e) The following are the conjugacy classes of A_4: $\{(1)\}$, $\{(12)(34), (13)(24), (14)(23)\}$, $\{(123), (142), (134), (243)\}$, $\{(132), (124), (143), (234)\}$.

Prove that A_4 does not have a subgroup of order 6. You may use any fact on normal subgroups proved in the course.

4. **(a)** Define what is meant by an action of a group G on a set X. For an element x of X define what is meant by the orbit $Orb(x)$ of x and what is meant by the stabiliser subgroup $St(x)$ of x. Assuming G is finite, state a formula relating $|G|$, $|St(x)|$ and $|Orb(x)|$.

Hence show that if a group of order 27 acts on a set with 56 elements, then there are at least two orbits of size 1.

(b) Let p be a prime. Define the notion of a Sylow p-subgroup of a finite group.

(i) State the first Sylow theorem.

(ii) Write down a Sylow 2-subgroup of A_4 and a Sylow 3-subgroup of S_5. You need not justify your answer.