UNIVERSITY OF ABERDEEN **SESSION 2005–06**

Degree Examination
MX3019 Mechanics and Set Theory
Thursday 19 January 2006 (9 am to 12 noon)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

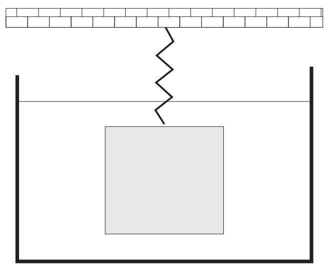
Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FIVE questions including no more than THREE from either section.
All questions carry equal weight.
Answer each section in a different answer book.

SECTION A (MX3012 Mechanics)

1. Coloumb used the following method for determining the fluidity coefficients of liquids. He attached a spring at the middle of one of the edges of a rectangular plate; the other end of the spring is fixed and the plate can oscillate in the vertical direction.

Coloumb measured the frequencies of oscillations of the plate, first in the air and then in the liquid. Let us denote these frequencies by ω_1 and ω_2, respectively. The magnitude of the friction force between the plate and the liquid can be expressed as $2Skv$, where $2S$ is the area of the plate, v is the magnitude of the velocity of the plate and k is the fluidity coefficient.



Fix a Cartesian coordinate system with the x-axis vertically down along the spring and put the origin in such a way that the x-coordinate describes the displacement of the plate from the position where the rigid force produced by the spring vanishes.

Neglecting the air resistance in the first case derive that the equation for $x(t)$ given by Newton's second law for the plate is

$$m\ddot{x} = mg - cx,$$

where c is the rigidity of the spring and m is the mass of the plate.

Neglecting buoyancy in the second case, prove that the equation of motion for the plate is

$$m\ddot{x} = mg - cx - 2Sk\dot{x}.$$

Assuming that the damping in the second case is light find the general solutions to the equations of motion in both cases and determine the corresponding frequencies of oscillations ω_1 and ω_2.

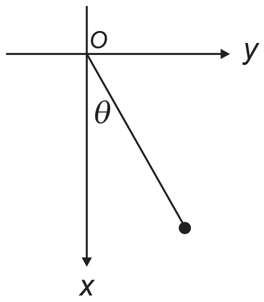
Express k in terms of ω_1, ω_2, m and S.

2. A pendulum consists of a thin rod of length l with a particle of mass m attached at one end. The other end of the rod is attached to a fixed point O in such a way that the pendulum can swing freely in a vertical plane through O (the particle moves in a circle of radius l about O). Frictional forces are neglected in this problem.

Taking the origin at O, choosing the x-axis vertically down and using polar coordinates, derive the equation of motion

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta), \tag{1}$$

where θ is the (oriented) angle swept out by the pendulum from the x-axis and g is acceleration due to gravity. What are the equilibrium points of the ODE (1)?



Show that the linear approximation to (1) at the point $\theta = 0$ is

$$\ddot{\phi} = -\frac{g}{l}\,\phi, \tag{2}$$

where $\phi \approx \theta$. Find the general solution of (2).

Find the solution of (2) which satisfies the initial conditions $\phi = 0$, $\dot{\phi} = \omega$ when $t = 0$.

3. **(a)** Define what is meant by the *angular momentum* (relative to the origin) of a particle of mass m, with position vector $\mathbf{r}$ and velocity vector $\dot{\mathbf{r}}$.

(b) Define what is meant by a *central force* $\mathbf{F}(\mathbf{r})$. State and prove the law of conservation of angular momentum for a particle of mass m moving under the influence of a central force.

(c) Define what is meant by a *conservative force.*

(d) Determine if the force $\mathbf{F}(\mathbf{r})$ is conservative (justify your answer) and, if it is, find a corresponding potential energy in the cases

(i) $\mathbf{F}(\mathbf{r}) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$,

(ii) $\mathbf{F}(\mathbf{r}) = zy\mathbf{i} + xz\mathbf{j} + (xy + e^z)\mathbf{k}$.

Here, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the orthonormal vectors of a Cartesian coordinate system, and x, y, z are the corresponding coordinates.

4. A particle of mass m is fired radially outwards from a point distance R from the origin with speed of magnitude u and moves under a gravity force $\mathbf{F}(\mathbf{r}) = -G\frac{Mm}{r^3}\mathbf{r}$, where G and M are constants, $\mathbf{r}$ is the radius-vector of the particle with respect to the given origin, r is the magnitude of $\mathbf{r}$.

Prove that the force $\mathbf{F}$ is conservative and find the corresponding potential energy.

Using the energy conservation law prove that if

$$u \geq \sqrt{\frac{2GM}{R}}$$

the particle will never return to the point of firing.

<u>SECTION B (MX3017 Set Theory)</u>

5. Let X be a set and A, B, C and D subsets of X.

(i) Show that if $A \subset B$ and $C \subset D$ then $A \cup C \subset B \cup D$.

(ii) Define the *complement* $\complement A$ of A in X and show that $A \subset B \Leftrightarrow \complement B \subset \complement A$.

(iii) Suppose Y is also a set and $f : X \to Y$ a function. Define the *power set* $P(Y)$ of Y and also the function $f^{-1} : P(Y) \to P(X)$. If E and F are subsets of Y, show that

$$f^{-1}(E) \cap f^{-1}(F) = f^{-1}(E \cap F).$$

(iv) With the same notation as above, define the expressions $f(A)$ and $A \setminus B$. Show that

$$f(A) \setminus f(B) \subset f(A \setminus B). \qquad (*)$$

Show also, by means of an example, that, in $(*)$, one cannot replace $\subset$ by equality.

6. (i) Define the terms *partition, relation, partial order* and *linear partial order* as they are applied in set theory.

(ii) Let X be a set and let $A, B \in P(X)$, where $P(X)$ denotes the power set of X. Define a relation R on $P(X)$ by $ARB \Leftrightarrow A \subset B$. Show that R is a partial order on $P(X)$. Show also, by choosing a particular example for X, that this partial order need not be linear.

(iii) Let A and B be non-empty sets and let f denote the map $A \times B \to A$ given by $f : (a, b) \to a$ for $a \in A$ and $b \in B$ (the *projection* of $A \times B$ onto A). Let P be a *partition* of A. Show that $\{f^{-1}(C) : C \in P\}$ is a partition of $A \times B$.

7. State, *without proof*, the recursion theorem. Show how this theorem may be used to define addition and multiplication in $\mathbb{N}$.

Using the symbols $+$ and . to denote the addition and multiplication operations defined above and n' to denote the *successor* of n, show that, for $m, n, k \in \mathbb{N}$, and with $o = \emptyset$,

(i) $n + o = n$

(ii) $m.o = o.m = o$

(iii) $m + n' = (m + n)'$

(iv) $(m + n) + k = m + (n + k)$

8. (i) Show that a subset of a countable set is countable.

(ii) Let A and B be countably infinite sets. For each $b \in B$ define the subset Ab of $A \times B$ by $Ab = \{(a, b) : a \in A\}$. Show that A and Ab are *equivalent* (i.e. $A \approx Ab$) for each $b \in B$. Then show that $A \times B$ is countably infinite.

(iii) Let A and B be infinite sets with A countable and B not countable. Are the sets $A \cap B$ and $A \cup B$ countable or not? Justify your answer. Give an example of this situation in which A and B are subsets of $\mathbb{R}$ and where $A \cap B = \emptyset$.

[In part (ii) you may assume, *without proof*, that a subset of a finite set is finite and that a countable union of countable sets is countable.]