UNIVERSITY OF ABERDEEN **SESSION 2006–07**

DEGREE EXAMINATION

MX3017 Set Theory

Tuesday 23 January 2007 (9 am to 11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. The questions are equally weighted.

1. Let X be a set and let A and B be subsets of X.

(a) Define the sets $A \setminus B$ and the *complement* $\complement A$ of A in X. Show that $\complement(\complement A) = A$.

(b) Show that $\complement(A \cap B) = \complement A \cup \complement B$ and that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(c) Suppose that $A \subset B$. Show that $A \cup C \subset B \cup C$ for *any* subset C of X and that $A \cap C \subset B \cap C$ for *any* subset C of X. State clearly the converses of these results and show, by making appropriate choices for C, that each converse statement is true.

2. **(a)** Define the terms *partition, relation, equivalence relation, equivalence class* and *function* as they are applied in set theory.

(b) Show that the set of equivalence classes of an equivalence relation on a set X is a partition of X.

(c) Let X and Y be sets and let $f : X \to Y$ be a function from X to Y. Let A be a subset of X and B be a subset of Y. Define the sets $f(A)$ and $f^{-1}(B)$ and the terms *one-to-one* and *onto* as applied to f. Show that f is onto if and only if $f(X) = Y$.

(d) Let X and Y be sets and $f : X \to Y$ a function from X to Y. If A and B are subsets of X show that

$$f(A \cup B) = f(A) \cup f(B)$$

Is it true that $f(A \setminus B) = f(A) \setminus f(B)$? You should either prove it true or supply a counterexample.

3. Starting with the set $\mathbb{N}$ of natural numbers, write an essay, including brief mathematical details, where necessary, describing the construction of the set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$. [A good answer would include a brief but informative discussion of the following steps; (i) go from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, put an equivalence relation on this latter set to get to $\mathbb{Z}$. (ii) Discuss how addition and multiplication may be defined on $\mathbb{Z}$. (iii) Construct $\mathbb{Q}$ by a similar procedure and discuss addition and multiplication on $\mathbb{Q}$. (iv) Given the usual ordering on $\mathbb{N}$ show how it can be extended to $\mathbb{Z}$ and $\mathbb{Q}$.]

4. **(a)** Define the following phrases and terms as they are applied to sets; *of the same cardinality as*, *finite*, *infinite*, *countable* and *countably infinite.* Show that if A and B have the same cardinality then A is finite if and only if B is finite and that A is infinite if and only if B is infinite.

(b) Using the fact that a set is finite if and only if it is not of the same cardinality as *any proper* subset of itself, show that $\mathbb{N}$ is countably infinite.

(c) Let $\mathbb{N}$ be the set of natural numbers. Show that the subset of $\mathbb{N}$, consisting of those positive integers which leave a remainder of 2 upon division by 5, is countably infinite.

(d) Using a result from set theory, which should *not* be proved but should be stated clearly without proof, or otherwise, show that the subset of $\mathbb{R}$ consisting of all *irrational* numbers is uncountable.

[Any results that you wish to use regarding successors of members of $\mathbb{N}$ and from the rules of addition and multiplication in $\mathbb{N}$, should be stated clearly, without proof].