UNIVERSITY OF ABERDEEN **SESSION 2003-04**

Degree Examination

MX3014 Inference and Modelling

Wednesday 21 January 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. Suppose that the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{11}(x^2+xy) & \text{for } 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (i) Find the marginal densities of X and Y.

(ii) Are X and Y independent? (Give reasons.)

(b) Find the conditional density of X, given $Y=y$ $(0<y<1)$.

(c) Show that $P(X<Y) = \dfrac{5}{88}$.

Hint: It may be helpful to sketch the region corresponding to the event $\{X<Y\}$.

2. **(a)** Let X follow the χ^2-distribution with v degrees of freedom (see formulae sheet or statistical tables for density function). Use suitable transformations to find the probability densities of the random variables

(i) $Y = X/v$,

(ii) $Z = \sqrt{X/v}$,

(iii) $W = \sqrt{X}$.

(b) The position of a particle has co-ordinates (X_1, X_2, X_3), where X_1, X_2, X_3 are independent $N(0,1)$ variables.

(i) Using the results of **(a)**, or otherwise, find the probability density function of the distance, D, of the particle from the origin.
Note: $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$.

(ii) Find the median of D.

3. A certain random variable X has probability density function

$$f(x|\lambda) = \frac{1}{24}\lambda^5 x^4 e^{-\lambda x} \quad (x > 0,\ 0 \text{ otherwise})$$

where λ is a positive parameter.

(a) Suppose that λ is known.

(i) Using the formulae sheet, or otherwise, write down the moment generating function, $M_X(s)$, of X.

Note: $\Gamma(x) = (x-1)!$ for positive integer values of x.

(ii) Using the moment generating function, or otherwise, find $E(X^3)$ in terms of λ.

(iii) Consider a random sample, $X_1, X_2, \ldots, X_n$, from this distribution.

(1) Find the moment generating function of $Y = X_1 + X_2 + \ldots + X_n$, and the moment generating function of $Z = 2\lambda Y$.

(2) Show that Z follows a χ^2-distribution, and state the number of degrees of freedom.

(b) Now suppose that λ is unknown ($\lambda > 0$). You are given n observations $x_1, x_2, \ldots, x_n$ of the variable X.

(i) Find the maximum likelihood estimate, $\hat{\lambda}$, of λ.

(ii) Assuming that n is large, find a formula for an approximate 95% confidence interval for λ in terms of $x_1, x_2, \ldots, x_n$ and n.

4. Let the variables Y_1, Y_2, Y_3, Y_4 be modelled by

$$\begin{aligned} Y_1 &= \beta_1 + \beta_2 + \varepsilon_1 \\ Y_2 &= 2\beta_1 - \beta_2 + \varepsilon_2 \\ Y_3 &= 2\beta_1 + \varepsilon_3 \\ Y_4 &= \beta_1 + 2\beta_2 + \varepsilon_4 \end{aligned}$$

where β_1, β_2 are unknown parameters and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are independently and identically distributed, each having mean zero and variance σ^2.

(a) Write down this model in terms of vectors and matrices, and hence find the least squares estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ (in terms of Y_1, Y_2, Y_3 and Y_4) and their variances and covariance.

(b) The observed responses are $y_1 = 4.2$, $y_2 = 7.3$, $y_3 = 8.5$, $y_4 = 6.2$. Find the least squares estimates $\hat{\beta}_1$ and $\hat{\beta}_2$, and calculate the residual sum of squares for this model.