UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION
MX3001 Real Analysis
Friday 21 January 2005 (12noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Determine all real numbers x that satisfy $\left|\dfrac{x+1}{2x-3}\right| < 2.$

(b) Let A be a non-empty subset of $\mathbb{R}$ which is bounded above. Define the supremum of A, $\sup(A)$.

Let (a_n) be a monotonically increasing sequence of real numbers which is bounded above. Show that the sequence (a_n) converges as n tends to infinity.

(c) The sequence (x_n) is defined inductively by

$$x_1 = 0, \qquad x_{n+1} = \frac{3x_n + 1}{x_n + 3} \quad (n \geq 1).$$

(i) Show, by induction, that $0 \leq x_n \leq 1$ for all $n \geq 1$.

(ii) Show that the sequence (x_n) is monotonically increasing.

(iii) Deduce that the sequence (x_n) converges and find the limit of the sequence.

2. **(a)** Determine if the sequence whose n-th terms are given below converges or diverges.

$$\text{(i)} \quad \frac{n^2+n}{3n^2-2}, \qquad \text{(ii)} \quad \frac{n^{3/2}}{\sqrt{n^2+3}}, \qquad \text{(iii)} \quad \frac{3^{n+1}-2}{2^{2n}+1}.$$

(b) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Define what is meant by the statement that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is *convergent*.

Determine if the following series are convergent:

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{n^{1/2}}{5n^3+1}, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{3^n}{n!}, \qquad \text{(iii)} \quad \sum_{n=1}^{\infty} \frac{3^n+1}{5^n+2}.$$

(c) Explain the meaning of *absolutely* convergent and *conditionally* convergent as applied to a series $\displaystyle\sum_{n=1}^{\infty} a_n$.

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ is convergent.

Is this series absolutely convergent or conditionally convergent? (A proof is not required.)

3. Let $f(x)$ be a real valued function defined on the interval (c, d), except possibly at the point $a \in (c, d)$.

(i) Define the notation $\lim_{x \to a} f(x) = k$.

(ii) Prove from your definition that $\lim_{x \to 2} (x^2 + x) = 6$.

What can be deduced about the continuity of $g(x) = x^2 + x$?

(iii) Let $\lim_{x \to a} f(x) = k > 0$. Prove that there exists $\delta_1 > 0$ such that $f(x) > \frac{k}{2}$ when $0 < |x - a| < \delta_1$.

Show also that $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{k}$.

4. **(a)** State the Intermediate Value Theorem.

Given a continuous function $f(x)$ defined on $[0, 1]$ and taking values in $[0, 1]$, show that there exists $x \in [0, 1]$ such that $f(x) = x$.

(b) Let $f(x)$ be a real valued function defined on an interval I. Define what is meant by $f(x)$ *is uniformly continuous on* I.

Prove that $f(x) = x^2 + x$ is uniformly continuous on $(-2, 2)$.

(c) Consider suitable partitions of $[0, 1]$ to show that the Riemann Integral $\int_0^1 x \, dx$ exists and its value is $\frac{1}{2}$.