UNIVERSITY OF ABERDEEN **SESSION 2003–04**

Degree Examination

MX3001 Real Analysis

Thursday 22 January 2004 (3pm to 5pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Determine all real x which satisfy $\dfrac{3-2x}{x+2} < 2$.

(b) For each positive integer n, let $h_n = n^{1/n} - 1$. By considering $(1+h_n)^n$, prove that if $n > 1$ then $0 < h_n^2 < \dfrac{2}{n-1}$. Deduce that $\lim_{n\to\infty} n^{1/n} = 1$.

(c) The sequence (a_n) is defined inductively by $a_1 = 0$, $a_{n+1} = \dfrac{2a_n+1}{2+a_n}$ for $n \geq 1$. Prove that (a_n) is a monotonically increasing sequence bounded above. Deduce that the sequence (a_n) converges and find its limit.

2. **(a)** Explain what is meant by the statement that the series $\sum_{n=1}^{\infty} a_n$ *converges.*

(b) Use the comparison tests to show that one of the following series is convergent and one is divergent:

$$\sum_{n=2}^{\infty} \frac{n-1}{n^2+4n}\,; \qquad \sum_{n=1}^{\infty} \frac{n^2-\sqrt{n}}{5n^4+\sqrt{n}}\,.$$

(c) State the ratio test for a series of positive terms.

Determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}\,; \qquad \sum_{n=1}^{\infty} \frac{3^n-2^n}{4^n+1}\,.$$

3. **(a)** Let A be a subset of $\mathbb{R}$ which is bounded above. State two properties of the completeness axiom which characterize the supremum of A, that is $\sup(A)$.

Let (a_n) be a monotonically increasing sequence that is bounded above. Prove that (a_n) is a convergent sequence.

(b) Let f be a real function defined on an open interval I and let $a \in I$. Define what is meant by $\lim_{x \to a} f(x) = k$.

Let $f(x) = \dfrac{2}{x-1}$, $x \neq 1$. Prove from first principles that $\lim_{x \to 2} f(x) = 2$.

(c) Let $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Show that $\lim_{x \to 0} \sin(\dfrac{1}{x})$ does not exist.

4. **(a)** Let $f(x)$ and $g(x)$ be real valued functions that are continuous at $a \in \mathbb{R}$.

Prove that there exists $\delta > 0$ such that $g(x)$ is bounded on the interval $(a - \delta,\ a + \delta)$.

Prove that the product function $f(x)g(x)$ is continuous at a. [Hint: $f(a)g(a) - f(x)g(x) = f(a)\,(g(a) - g(x)) + g(x)\,(f(a) - f(x))$.]

(b) Let $f(x)$ be a real valued function defined on an interval I. Define what is meant by $f(x)$ is *uniformly continuous* on I.

Prove that $f(x) = x^2 - x$ is uniformly continuous on the interval $(-1, 1)$.

State reasons why $g(x) = \cos\left(\sin(e^{x^2+1})\right)$ is uniformly continuous on $[-7, 2]$.