UNIVERSITY OF ABERDEEN **SESSION 2006–07**

DEGREE EXAMINATION
MX3001 Real Analysis
Tuesday 16 January 2007 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Let $A = \{\frac{x-1}{x+3} \ : x \in \mathbb{R}, x \geq 0\}$. Prove that $\sup(A) = 1$ and $\inf(A) = -\frac{1}{3}$.

(b) Let (a_n) be a real sequence and let l be a real number. Define what is meant by the statement $a_n \to l$ as $n \to \infty$.

(i) Suppose that (a_n) is a real sequence converging to l, $l \in \mathbb{R}$. Prove that there is a number K such that $K < a_n$ for all $n \in \mathbb{N}$.

(ii) Suppose that (a_n) and (b_n) are real sequences converging to l and m respectively, $l, m \in \mathbb{R}$. Prove that the sequence $(a_n - b_n)$ converges to $l - m$.

(c) State the squeezing lemma for real sequences.

Prove that $\frac{2^{n+1}}{n!} \to 0$ as $n \to \infty$.

2. **(a)** (i) Define what it means for a series $\sum_{n=1}^{\infty} a_n$ to be convergent

Prove that the series $\sum_{n=1}^{\infty} \frac{4}{n(n+1)}$ is convergent.

(ii) State the second comparison test for convergence of positive series.

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(b) Determine if the following series are convergent or divergent. State clearly the names of any tests used, and show how these are being used.

$$\sum_{n=1}^{\infty} \frac{n-2}{n^3+1}, \quad \sum_{n=1}^{\infty} \frac{2+5^n}{1+6^n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

3. **(a)** Let $f(x)$ be a real valued function defined on the interval (c, d) except possibly at the point $a \in (c, d)$. Define the notation $\lim_{x \to a} f(x) = k$.

Prove from your definition that $\lim_{x \to 1} \frac{1}{x+2} = \frac{1}{3}$.

(b) Let $f(x) : [a, b] \to \mathbb{R}$ be a continuous function. Let $M = \inf\{f(x) \ : \ x \in [a, b]\}$. Show that there exists $c \in [a, b]$ such that $f(c) = M$. You may use the following fact:

If $g(x) : [a, b] \to \mathbb{R}$ is continuous, then $g(x)$ is bounded on $[a, b]$. You also may use any result on continuity resulting from the algebra of limits for real valued functions.

(c) State the Intermediate Value Theorem.

Show that there exists a solution to the following equation $x^3 - x^2 - 2x + 1 = 0$.

4. **(a)** Let $f(x)$ be a real valued function defined on an interval I. Define what is meant by $f(x)$ *is uniformly continuous on* I.

Show that $x^2 + 1$ is uniformly continuous on $(-2, 2)$.

(b) Give an example of an open interval I and a real valued function $f(x)$ defined on I such that $f(x)$ is continuous on I but $f(x)$ is not uniformly continuous on I. Justify your example.

(c) Consider suitable partitions of $[0, 1]$ to show that the Riemann Integral $\int_0^1 (2x + 1)dx$ exists and that its value is 2.