UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination

MA2507 Advanced Calculus

Friday 25 May 2007 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FOUR questions. All questions carry equal weight.

1. **(a)** Suppose that

$$f(x,y) = 1 + \frac{5x^2y^3}{x^2+y^2} \qquad \text{for} \qquad (x,y) \neq (0,0)$$

and that $f(0,0) = 0$. By applying the Squeezing Rule to $|f(x,y) - 1|$, or otherwise, prove that $f(x,y) \to 1$ as $(x,y) \to (0,0)$. Give clear reference to any standard results that you use.

Is f continuous at $(0,0)$? Explain your answer. Also, explain briefly (by reference to standard results) why f is continuous at all points $(x,y) \neq (0,0)$.

(b) Suppose that

$$f(x,y) = 1 + x^2 - y^2 + \frac{2x^3 - 4y^3}{x^2+y^2} \qquad \text{for} \qquad (x,y) \neq (0,0)$$

and that $f(0,0) = 1$. By using the usual laws of partial differentiation, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points $(x,y) \neq (0,0)$. [You do not need to simplify your answers.]

For $h \neq 0$, simplify the expression

$$\frac{f(h,0) - f(0,0)}{h}.$$

Hence show that $\dfrac{\partial f}{\partial x}(0,0) = 2$. Find also $\dfrac{\partial f}{\partial y}(0,0)$.

2. **(a)** Let $f(x,y) = 2x^3 - 6x^2 + 6xy - 3y^2 - 4$. Find the critical points of f and classify each as a local maximum, a local minimum or a saddle point. Show that f does not achieve a global minimum.

In one sentence, give the name of a theorem from one-variable analysis which can be used in the proof of the second derivative test for functions of two variables.

(b) Suppose that $z = F(x,y)$, and let $s = xy$ and $t = \dfrac{x}{y}$ (for $x, y > 0$). Show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s} y + \frac{\partial z}{\partial t} \frac{1}{y}$$

and obtain a similar expression for $\dfrac{\partial z}{\partial y}$.

Hence show that this change of variables reduces the partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4x^2$$

to the form

$$\frac{\partial z}{\partial s} = 2t.$$

Deduce that $z = 2x^2 + g\left(\dfrac{x}{y}\right)$ where g is an arbitrary differentiable function of one variable.

Suppose further that $F(x,1) = 0$ for all $x > 0$. Find z in this case.

3. **(a)** Find the volume of the region under the surface $z = xy + 3xy^2$ and above the rectangle

$$R = \{(x,y) : 1 \leq x \leq 2,\ 0 \leq y \leq 1\}.$$

(b) Let D be the semi-circular region of radius R (> 0) given by

$$x^2 + y^2 \leqslant R^2, \qquad x \geq 0.$$

By using polar coordinates, or otherwise, evaluate

$$\iint_D x \, dx \, dy.$$

Hence show that the x-coordinate of the centroid of D is $\dfrac{4R}{3\pi}$.

(c) Sketch the region of integration for

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy.$$

Change the order of integration, and hence evaluate the integral.

4. Find the general solution of each of the following ordinary differential equations:

(i)

$$\frac{dy}{dx} + 2y = 3e^x,$$

(ii)

$$\frac{dy}{dx} = \frac{2xe^y}{1+x^2},$$

(iii)

$$\frac{dy}{dx} = 2\frac{y}{x} + x^3e^x,\ x > 0.$$

5. **(a)** Find the solution of the initial value problem:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0,$$
$$y = 1,\ \frac{dy}{dx} = 0 \text{ when } x = 0.$$

(b) Find the general solution of the inhomogenous ordinary differential equation:

$$2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 3y = e^{-x/2}.$$