UNIVERSITY OF ABERDEEN **SESSION 2005-2006**

Degree Examination
MX4037 Ordinary Differential Equations
Wednesday 18 January 2006 (3 pm to 5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions

1. **(a)** Let n be a positive integer, let I be a non-empty open interval in $\mathbb{R}$ and let A be a real $n \times n$-matrix valued map which is defined and continuous on I. Let $\psi_1, \psi_2, \ldots, \psi_n$ be a fundamental system of (real) solutions on I for the system

$$\dot{\mathbf{x}} = A(t)\,\mathbf{x}.$$

Let $s \in I$. Show that the vectors $\psi_1(s), \psi_2(s), \ldots, \psi_n(s)$ are linearly independent (over $\mathbb{R}$).

Show that every solution of the system on I can be expressed as a linear combination of ψ_1, $\psi_2, \ldots, \psi_n$.

[An appropriate uniqueness theorem may be used without proof.]

(b) Find the general solution of the system

$$\begin{aligned}\dot{x} &= 3x - y\\ \dot{y} &= -x + 3y.\end{aligned}$$

Sketch some integral curves of the system paying particular attention to the behaviour of the curves near to the origin.

2. **(a)** Let $\beta_1(t) = t^{-5}$ and $\beta_2(t) = t^{-1}$ for $t > 0$. Show that the functions β_1 and β_2 form a fundamental system of solutions on $(0, \infty)$ for the homogeneous equation associated to the normal form of the ordinary differential equation

$$t^2\,\ddot{y} + 7t\,\dot{y} + 5\,y = t^{-5} \qquad (t > 0). \tag{1}$$

Find the solution of (1) which satisfies the initial conditions $y = \dot{y} = 0$ when $t = 1$.

(b) Write down the general real solution of the ordinary differential equation:

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$$

3. **(a)** Let n be a positive integer and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. Let $\phi : \mathbb{R} \to \mathbb{R}^n$ be a solution of the system

$$\dot{\mathbf{x}} = f(\mathbf{x}).$$

Let $c \in \mathbb{R}$ and let $\chi : \mathbb{R} \to \mathbb{R}^n$ be defined by $\chi(t) = \phi(t - c)$ for all $t \in \mathbb{R}$. Show that χ is a solution of the system.

Let $\psi : \mathbb{R} \to \mathbb{R}^n$ be another solution of the system. Show that if $\psi(r) = \phi(s)$ where r and s are real numbers then $\psi(t) = \phi(t - (r - s))$ for all $t \in \mathbb{R}$.

Deduce that the integral curves corresponding to the two solutions ϕ and ψ are either identical or disjoint.

(b) Find the equilibrium point of the system

$$\begin{aligned} \dot{x} &= x^2 - y \\ \dot{y} &= -x - xy. \end{aligned}$$

Use the Principle of Linearised Stability to determine the stability (or otherwise) of the equilibrium point.

4. Find and classify the equilibrium points of the system

$$\begin{aligned} \dot{x} &= x - y \\ \dot{y} &= x - y^2. \end{aligned}$$

Sketch the integral curves of the system paying particular attention to the behaviour of the integral curves near any equilibrium point.