UNIVERSITY OF ABERDEEN **SESSION 2006–2007**

Degree Examination

MX 4037 Ordinary Differential Equations

Monday 15 January 2007 (9 am to 11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions

1. **(a)** Let n be a positive integer, let I be a non-empty open interval in $\mathbb{R}$, let $s \in I$ and let A be a (real) $n \times n$-matrix valued map which is defined and continuous on I.

Suppose that $\psi_1, \psi_2, \ldots, \psi_n$ form a fundamental system of solutions on I of the homogeneous linear system

$$\dot{\mathbf{x}} = A(t)\,\mathbf{x}.$$

Show that the vectors $\psi_1(s)$, $\psi_2(s)$, $\ldots$, $\psi_n(s)$ are linearly independent.

Show that if ψ_1, ψ_2, $\ldots$, ψ_n are linearly independent then every solution of the system on I can be expressed as a linear combination of ψ_1, ψ_2, $\ldots$, ψ_n.

[An appropriate uniqueness theorem may be used without proof.]

(b) Find a fundamental system of solutions for the system

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 4x - y. \end{aligned}$$

Sketch some integral curves of the system paying particular attention to the behaviour of the curves near to the origin.

2. **(a)** Let $\beta_1(t) = t$ and $\beta_2(t) = t^2$. Show that β_1 and β_2 form a fundamental system of solutions on $(0, \infty)$ for the normal form of the homogeneous equation

$$t^2\,\ddot{y} - 2t\,\dot{y} + 2\,y = 0.$$

Solve the initial value problem:

$$t^2\,\ddot{y} - 2t\,\dot{y} + 2\,y = t, \quad y = 1,\ \dot{y} = 1 \text{ when } t = 1.$$

(b) Solve the initial value problem:

$$\dddot{y} - \ddot{y} - \dot{y} + y = 0, \quad y = 1,\ \dot{y} = 0,\ \ddot{y} = 0 \text{ when } t = 0.$$

[Note that 1 is a root of the characteristic equation of the ordinary differential equation.]

3. **(a)** Let n be a positive integer and let f be a C^1-map from $\mathbb{R}^n$ into $\mathbb{R}^n$ and consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}).$$

(i) Let $\phi : \mathbb{R} \to \mathbb{R}^n$ be a solution of the system. Suppose that there are real numbers r, s ($r \neq s$) such that $\phi(r) = \phi(s)$. Show that ϕ is a periodic solution of the system.

(ii) Define what is meant by an equilibrium point of the system.

Define what is meant by saying that an equilibrium point of the system is stable. Define what is meant by saying that an equilibrium point of the system is asymptotically stable.

In the case $f(\mathbf{x}) = A\mathbf{x}$ where A is an $n \times n$ real matrix with a positive (real) eigenvalue, prove, from first principles, that the origin is not a stable equilibrium point.

(b) Find the equilibrium point of the system

$$\begin{aligned} \dot{x} &= x + y + 1 \\ \dot{y} &= (x - y)^2 \end{aligned}$$

and determine the stability (or otherwise) of the equilibrium point.

4. Find and classify the equilibrium points of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y + 2x - x^3. \end{aligned}$$

Sketch some integral curves of the system paying particular attention to the behaviour of the integral curves near to any equilibrium point.