UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION

MX3012 Mechanics A

Tuesday 18 January 2005 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. A bullet is fired over horizontal ground from a point O on the ground at time $t = 0$. In motion the only forces on the bullet are gravity and air resistance. The air resistance has magnitude proportional to the speed of the bullet and direction opposite to the velocity of the bullet. Assuming that the acceleration g due to gravity is constant, show that the equation of motion of the bullet can be expressed as

$$\ddot{\mathbf{r}} + \beta\dot{\mathbf{r}} = -g\mathbf{k}$$

where $\mathbf{r}$ is the position vector of the bullet relative to O, $\mathbf{k}$ is the unit vector with direction vertically upwards and β is a positive constant. Deduce that

$$\dot{\mathbf{r}} + \beta\mathbf{r} = -gt\mathbf{k} + \mathbf{u}$$

where $\mathbf{u}$ is the velocity of the bullet at the instant of firing.

Show that the height z of the bullet at time t is given by

$$z = \frac{g}{\beta^2}(1 - \beta t - e^{-\beta t}) + \frac{1}{\beta}(1 - e^{-\beta t})\mathbf{u}\cdot\mathbf{k}.$$

2. A particle of mass m is moving in the xy-plane round the circle with centre at the origin O and radius a (a is a positive constant). The position vector $\mathbf{r}$ of the particle relative to O can be expressed as

$$\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}$$

where the variable θ depends on time t. Let $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$. Derive expressions for the velocity and acceleration of the particle relative to O in terms of a, $\mathbf{e}_r$, $\mathbf{e}_\theta$, θ and derivatives of θ with respect to t.

The resultant force on the particle is the sum of a central force $\mathbf{T}$ with centre at O and a force due to friction. While the particle moves, the frictional force has *constant* magnitude k and has direction *opposite* to the velocity of the particle. Initially the speed of the particle is u. Both k and u are positive. How long will it take for the speed to halve? [Give your answer in terms of k, u, a and m.]

Determine the magnitude of the central force $\mathbf{T}$ as a function of t for $t < \dfrac{mu}{k}$.

3. **(a)** A particle slides smoothly along a surface. The forces on the particle are the reaction from the surface and a conservative force. Use D'Alembert's Principle to show that the total energy of the particle is conserved.

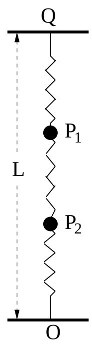
(b) A bead of mass m slides smoothly on the plane with equation $x + 2y - z = 0$ under the influence of gravity. (Here x, y and z are coordinates relative to an Earth frame whose z axis is vertically upwards.) At time $t = 0$ the particle is released from rest at the point with coordinate vector (1, 1, 3). Determine the coordinate vector of the particle at time $t = 10$. [You should assume that the acceleration due to gravity, g, is constant.]

4. **(a)** Consider a system consisting of n distinct particles P_1, P_2, …, P_n, with masses m_1, m_2, …, m_n and position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_n$, relative to the origin O of an inertial frame, respectively. For each $i = 1, 2, \ldots, n$, suppose that the only forces acting on P_i are internal forces $\mathbf{F}_{ij}$, which always act along the line joining P_i and P_j, for $j = 1, 2, \ldots, n$ and $j \neq i$.

Assume that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ and $i \neq j$.

Define what is meant by the *total angular momentum* of the system about O. Show, as a consequence of Newton's second law, that the total angular momentum of the system about O is constant.

(b) Two particles P_1 and P_2 of equal mass m are connected to springs as in the diagram below.



The springs are identical, each having natural length l and modulus λ. The fixed point Q is at a height L vertically above the fixed point O ($L > 3l$). Take the origin of coordinates at O and the z-axis vertically upwards and let c be the z-coordinate of the centre of mass of the particles.

The particles are in motion. Suppose that the particles remain on the z-axis throughout the motion. Show that c satisfies the equation

$$\ddot{c} = \left(\frac{\lambda L}{2lm} - g\right) - \frac{\lambda}{lm}\, c.$$

[You should assume that acceleration due to gravity, g, is constant and that the springs obey Hooke's law throughout the motion.]

Write down the general solution of the equation.