UNIVERSITY OF ABERDEEN **SESSION 2002–03**

DEGREE EXAMINATION

MX3012 Mechanics A

Friday 23 January 2004 (3pm to 5pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. A bead P of mass m slides without friction along a string which occupies the line $y = 0$, $z = 1$ in 3–space. A force $-m\mathbf{r}/\|\mathbf{r}\|^3$ acts on P, where $\mathbf{r} = x\mathbf{i} + \mathbf{k}$ is the position vector of P. There are no other external forces.

(i) Using d'Alembert's principle, set up a second order differential equation for the function $t \mapsto x(t)$ which describes the x–coordinate of the particle as a function of time t.

(ii) Find the constant (time independent) solutions of the differential equation.

(iii) There is a smallest positive number c such that, if $x(0) = 0$ and $\dot{x}(0) > c$, then the bead will "escape" to infinity (that is, $\lim_{t\to+\infty} x(t) = +\infty$). Find c.

2. **(a)** Define what is meant by the *angular momentum* (relative to the origin) of a particle of mass m, with position vector $\mathbf{r}$ and velocity vector $\dot{\mathbf{r}}$.

(b) Define what is meant by a *central force* $\mathbf{F}(\mathbf{r})$. State and prove the law of conservation of angular momentum for a particle of mass m moving under the influence of a central force.

(c) Define what is meant by a *conservative force* $\mathbf{F}(\mathbf{r})$. State and prove the law of conservation of (total) energy for a particle of mass m moving under the influence of a conservative force.

(d) Determine if the force $\mathbf{F}(\mathbf{r})$ is conservative (justify your answer) and, if it is, find a potential function in the cases

(i) $\mathbf{F}(\mathbf{r}) = yz\mathbf{i} + xz\mathbf{j} + 2xy\mathbf{k}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;

(ii) $\mathbf{F}(\mathbf{r}) = \dfrac{1}{\|\mathbf{r}\|}(-y\mathbf{i} + x\mathbf{j})$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

3. A bead of mass m slides without friction along the plane curve K given by $y = \sin x$. The only external force acting on the bead is $-mg\,\mathbf{j}$, due to gravity. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ be the position vector of the bead at time t.

(i) Use the relationship $y(t) = \sin(x(t))$ to find an expression for $\ddot{y}(t)$ in terms of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$.

(ii) Write down a nonzero tangent vector $\mathbf{v}(\mathbf{r})$ to the curve K at the point with position vector $\mathbf{r} = x\mathbf{i} + \sin(x)\mathbf{j}$.

(iii) Explain briefly what *d'Alembert's principle* means for the sliding bead. Hence, and using your result from part (ii), show that

$$\ddot{x} + \ddot{y}\cos x + g\cos x = 0 \qquad \text{for all } t.$$

(iv) From (i) and (ii), obtain a second order differential equation for the function $t \mapsto x(t)$; that is, an expression for $\ddot{x}(t)$ in terms of $x(t)$ and $\dot{x}(t)$. Do not attempt to solve the differential equation.

4. **(a)** Let P_1, P_2, $\ldots P_n$ be distinct particles in $\mathbb{R}^3$, with masses $m_1, m_2, \ldots, m_n$ and position vectors $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\ldots$, $\mathbf{r}_n(t)$ at time t. Suppose that the total force $\mathbf{F}_i^{\text{tot}}$ acting on P_i is the sum of an external force $\mathbf{F}_i^{\text{ext}}$ and interactive forces $\mathbf{F}_{ij}^{\text{int}}$ for $j = 1, \ldots, n$. (These forces may depend on the position of all the particles, their velocity and the time t.) Assume $\mathbf{F}_{ij}^{\text{int}} = -\mathbf{F}_{ji}^{\text{int}}$, $\mathbf{F}_{ii}^{\text{int}} = \mathbf{0}$, and that $\mathbf{F}_{ij}^{\text{int}}$ is always parallel to the line through P_i and P_j.

(i) Define what is meant by the *centre of mass* of the system of particles $P_1, \ldots, P_n$. Using Newton's second law, show that

$$\Big(\sum_{i=1}^{n} m_i\Big)\,\ddot{\mathbf{q}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}$$

where $\mathbf{q}$ is the position vector of the centre of mass.

(ii) Define what is meant by the *angular momentum vector* (relative to the origin) of the system of particles $P_1, \ldots, P_n$. Using Newton's second law, show that the angular momentum vector is constant as a function of time t if all external forces $\mathbf{F}_i^{\text{ext}}$ are zero.

(b) Two particles P_1 and P_2 move along the (horizontal) x–axis, with x-coordinates $r_1(t)$ and $r_2(t)$ at time t, respectively. Both have mass 1. Particle P_1 is attached to the right–hand end of a spring whose left–hand end is attached to the origin. Particle P_2 is attached to the right–hand end of a spring whose left–hand end is attached to P_1. Both springs have natural length 1 and spring constant 1. (The particles are not influenced by any gravitational forces.) Draw a picture of the situtation. Using Hooke's law and Newton's second law, write down a system of two second order differential equations for the functions $t \mapsto r_1(t)$ and $t \mapsto r_2(t)$. Do not attempt to solve the system.