UNIVERSITY OF ABERDEEN **SESSION 2006–07**

DEGREE EXAMINATION

MX 3012 Mechanics A

Wednesday 17 January 2007 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. A particle of mass m is fired vertically upwards from the ground with speed of magnitude u. The only forces acting on the particle are gravity and an air-resistance that directly opposes the motion of the particle and has magnitude αv^2, where v is the magnitude of the speed of the particle and α is a positive constant.

Take the origin at the point of firing and let the x-axis point vertically upwards. Write down the equation of motion of the particle. Use the equation of motion to obtain the following equation for $\dot{x}$ as a function of x:

$$\dot{x}\frac{d\dot{x}}{dx} = -(g + \beta\dot{x}^2), \text{ where } \beta = \alpha/m. \tag{1}$$

Find the general solution of (1) and use it and the initial conditions to calculate $\dot{x}$ as a function of x. Deduce that the greatest height H reached by the particle is given by

$$H = \frac{m}{2\alpha}\ln(1 + \frac{\alpha}{mg}u^2).$$

Now assume that there is no air resistance. Using the energy conservation law find the greatest height H_0 reached by the particle in that case.

2. A particle P of mass m hangs from a support S on the end of a spring of natural length l and modulus λ. Show that when P is in the equilibrium position the length of the spring is $\frac{mgl}{\lambda} + l$. Choose the origin at P (when P is in the equilibrium position) and measure x downwards. Assume that the particle P is released from rest when the distance from S to P is equal to l.

Show that the motion of P is described by the following equation

$$\ddot{x} + \omega^2 x = 0, \text{ where } \omega^2 = \frac{\lambda}{ml}. \tag{2}$$

Using the general solution to equation (2) and the initial conditions show that

$$x(t) = -\frac{mgl}{\lambda}\cos\omega t. \tag{3}$$

Using solution (3) find the distance $d(t)$ from the support S to P as a function of t. Find those values of t for which the distance $d(t)$ is maximal.

3. A particle of mass m collides elastically with a particle of mass M (with $M > m$) which is initially at rest. The first particle leaves the collision at an angle $\frac{\pi}{2}$ with its initial direction.

Write down the equations of conservation of momentum and kinetic energy for the collision.

Show that

$$\cos\beta = \frac{m}{M}\frac{u}{w},$$

where β is the angle between the direction of the second particle after the collision and the initial direction of the first particle, u is the magnitude of the velocity of the first particle before the collision, and w is the magnitude of the velocity of the second particle after the collision.

Using the energy and the momentum conservation laws show also that

$$w = \sqrt{\frac{2m^2}{M(M+m)}}u.$$

Hence deduce that

$$\cos\beta = \sqrt{\frac{1}{2}(1+\frac{m}{M})}.$$

Using the identity $2\cos^2\beta - 1 = \cos 2\beta$ obtain the following expression for the angle β:

$$\beta = \frac{1}{2}\arccos\frac{m}{M}.$$

4. **(a)** Define what is meant by the work W done on a particle by a force $\mathbf{F}$.

(b) Define what is meant by a conservative force and by a potential energy of a particle moving in a conservative force field.

(c) Determine if the force $\mathbf{F}(\mathbf{r})$ is conservative and if it is, find a corresponding potential energy in the following cases

(i) $\mathbf{F}(\mathbf{r}) = (yz^2 + z)\mathbf{i} + xz^2\mathbf{j} + (2xyz + x)\mathbf{k}$

(ii) $\mathbf{F}(\mathbf{r}) = zy\mathbf{i} + xz\mathbf{j} - xy\mathbf{k}$

Here $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the orthonormal vectors of a Cartesian coordinate system and x, y, z are the corresponding coordinates.

(d) Define what is meant by the kinetic energy of a particle.

(e) State and prove the energy conservation law for a particle moving under the action of a potential force.