UNIVERSITY OF ABERDEEN **SESSION 2005–06**

Degree Examination
MA3018 Analysis and Algebra 1
Wednesday 18 January 2006 (9 am to 12 noon)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FIVE questions including no more than THREE from either section.
All questions carry equal weight.
Answer each section in a different answer book.

SECTION A (MX3001 Real Analysis)

1. **(a)** Let $A = \left\{ \dfrac{3x+1}{x+2} : x \in \mathbb{R}, x \geq 0 \right\}$. Prove that $\mathrm{Sup}(A) = 3$ and $\mathrm{Inf}(A) = \frac{1}{2}$.

(b) Let (a_n) be a real sequence and let l be a real number. Define what is meant by the statement $a_n \to l$ as $n \to \infty$.

(i) Suppose that (a_n) is a real sequence converging to l, where $l \in \mathbb{R}$. Prove that there is a number $K > 0$ such that $|a_n| < K$ for all $n \in \mathbb{N}$.

(ii) Suppose that (a_n) and (b_n) are real sequences converging to l and m respectively, where $l, m \in \mathbb{R}$. Prove that the sequence $(a_n b_n)$ converges to lm.

(c) State the squeezing lemma for real sequences.

Prove that $\dfrac{3^n}{n!} \to 0$ as $n \to \infty$.

2. **(a)** (i) Define what it means for a series $\displaystyle\sum_{n=1}^{\infty} a_n$ to be convergent.

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

(ii) State the second comparison test for convergence of positive series.

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(b) Determine if the following series are convergent or divergent. State clearly the names of any tests used, and show how they are being used.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3+1}, \qquad \sum_{n=1}^{\infty} \frac{2^n+5^n}{6^n+1}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n+1}.$$

3. **(a)** Let $f(x)$ be a real valued function defined on the interval (c, d) except possibly at the point $a \in (c, d)$. Define the notation $\lim_{x \to a} f(x) = k$.

Prove from your definition that $\lim_{x \to 1} \dfrac{2}{x-3} = -1$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $M = \mathrm{Sup}\{f(x) \; : \; x \in [a, b]\}$. Show that there exists $c \in [a, b]$ such that $f(c) = M$. You may use the following fact: If $g : [a, b] \to \mathbb{R}$ is continuous, then g is bounded on $[a, b]$. You also may use any result on continuity resulting from the algebra of limits for real valued functions.

(c) State the Intermediate Value Theorem.

Show that there exists a real number x such that $\cos(2x) = x$.

4. **(a)** Let $f(x)$ be a real valued function defined on an interval I. Define what is meant by saying that $f(x)$ is **uniformly continuous** on I.

Show that $x^2 - 1$ is uniformly continuous on $(-2, 2)$.

(b) Give an example of an interval I and a real valued function $f(x)$ defined on I such that $f(x)$ is continuous on I but $f(x)$ is not uniformly continuous on I. Justify your example.

(c) Consider suitable partitions of $[0, 1]$ to show that the Riemann Integral $\int_0^1 (x + 3)dx$ exists and that its value is $\frac{7}{2}$.

5. **(a)** Define the term *subring* of a ring R. Show that the subset

$$\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$$

of $\mathbb{C}$ is a subring of $\mathbb{C}$.

(b) Define the term *field*. Show that the subset

$$\mathbb{Q}(\sqrt{-2}) = \{a + b\sqrt{-2} \mid s, t \in \mathbb{Q}\}$$

of $\mathbb{C}$ is a field. (You may assume that $\mathbb{Q}(\sqrt{-2})$ is a subring of $\mathbb{C}$.)

(c) Let R be a commutative ring. Let $b, a \in R$ be two elements of R, which are not both zero. Define what is meant by saying that an element $d \in R$ is a *greatest common divisor* of b and a (that is, that $d = \gcd(b, a)$).

(d) Consider the standard norm function

$$N : \mathbb{Z}[\sqrt{-2}] \longrightarrow \mathbb{Z}^+$$

defined by $N(a+b\sqrt{-2}) = a^2 + 2b^2$. You may assume that $\mathbb{Z}[\sqrt{-2}]$ together with this norm function forms a Euclidean domain.

In $\mathbb{Z}[\sqrt{-2}]$, let $b = 8 + 4\sqrt{-2}$ and $a = 4 - \sqrt{-2}$. Use the division algorithm to find $q, r \in \mathbb{Z}[\sqrt{-2}]$ such that $b = qa + r$ with $N(r) < N(a)$. Check that $N(r) < N(a)$ does in fact hold. Find $\gcd(b, a)$.

(e) Let A be a Euclidean domain with norm function ω. Let $a, b \in A$ be two elements which are not both zero. The division algorithm says there exists elements $q, r \in A$ such that $b = qa + r$ where either $r = 0$ or $\omega(r) < \omega(a)$. Let $d = \gcd(b, a)$ and $c = \gcd(a, r)$.

(i) Prove that $d \mid c$.

(ii) Prove that $c \mid d$.

(iii) Conclude that d and c are associates.

6. **(a)** Let A be an integral domain. Define what is meant by an *irreducible* element in A, and what is meant by a *prime* element in A.

(b) Consider the integral domain $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ with the norm function $N(a + b\sqrt{-5}) = a^2 + 5b^2$. You may assume that $a + b\sqrt{-5}$ is a unit if and only if $N(a + b\sqrt{-5}) = 1$. Observe that

$$(3 + \sqrt{-5})(3 - \sqrt{-5}) = 14 = 2 \cdot 7.$$

(i) Show that $3 + \sqrt{-5}$ is irreducible.

(ii) Show that $3 + \sqrt{-5}$ is not prime.

(iii) Assuming that $3 - \sqrt{-5}$, 2, and 7 are also irreducible, briefly explain why $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean domain.

(c) Let A be an integral domain. Suppose $q \in A$ has the property that the principal ideal (q) is a prime ideal.

(i) Prove that q is a prime element of A.

(ii) Given (i), prove that q is an irreducible element of A.

7. **(a)** Define the terms *ring homomorphism* and the *kernel* of a ring homomorphism.

(b) A map

$$\varphi : \mathbb{Z}[i] \longrightarrow \mathbb{F}_5$$

is defined by $\varphi(a + bi) = a + 3b \pmod 5$. Show that φ is a ring homomorphism.

(c) Define the term *ideal* in a commutative ring.

(d) Let R and S be commutative rings, and suppose $\varphi : R \longrightarrow S$ is a ring homomorphism. Prove that the kernel of φ is an ideal of R.

(e) Let

$$\varphi : \mathbb{Z}[i] \longrightarrow \mathbb{F}_5$$

be the ring homomorphism in part (b). Find the kernel of φ, expressing it as a principal ideal. Justify your answer.

8. **(a)** State *Eisenstein's Test*.

(b) Determine whether the following polynomials are irreducible in $\mathbb{Q}[x]$:

(i) $x^3 + 6x^2 + 9x - 12$,

(ii) $x^3 + 6x^2 + 9x - 9$.

(c) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial. Briefly explain why the quotient ring $\mathbb{Q}[x]/(f(x))$ is a field.

(d) Suppose $f(x) \in \mathbb{Q}[x]$ is not an irreducible polynomial. That is, $f(x) = g(x) \cdot h(x)$ where neither $g(x)$ nor $h(x)$ are units in $\mathbb{Q}[x]$. Let $I = (f(x))$.

(i) Show that $(g(x) + I)(h(x) + I) = 0 + I$.

(ii) Show that $g(x) + I \neq 0 + I$ and $h(x) + I \neq 0 + I$.

(iii) Prove that the quotient ring $\mathbb{Q}[x]/I$ is *not* a field.

(e) Let $f(x) \in \mathbb{F}_3[x]$ be defined by $f(x) = 2x^2 + 2$. You may assume that $f(x)$ is irreducible. Use the polynomial $f(x)$ to construct, as a quotient ring, a field $\mathbb{K}$ with 9 elements. Write a list of the elements in $\mathbb{K}$. Find the square of each nonzero element in $\mathbb{K}$.