UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination

MA2506 Linear Algebra

Thursday 24 May 2007 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define the concept of a *basis* of a K-vector space. Give an example of a basis of $K_2[t]$.

(Recall that $K_2[t]$ is the K-vector space of all polynomials of degree less than or equal to 2, with coefficients in K.)

(b) Find bases for the column space $\mathrm{Col}(A)$ and null space $N(A)$ of the real matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -3 & 2 & 1 \\ 3 & -2 & 10 & 4 & -1 \end{bmatrix}$$

(c) Consider the following ordered bases for $\mathbb{F}_5^3$

$$\mathcal{A} = (\,(4,\ 1,\ 3),\ (1,\ 1,\ 1),\ (3,\ 1,\ 0)\,) \quad \text{and} \quad \mathcal{B} = (\,(1,\ 2,\ 3),\ (0,\ 4,\ 1),\ (2,\ 2,\ 2)\,)\ .$$

Find the transition matrix $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{A}}$ from the basis $\mathcal{A}$ to the basis $\mathcal{B}$.

2. **(a)** Define the concept of a *linear transformation* $T\colon V \to W$, where V and W are vector spaces over a field K. Define the *kernel*, $\mathrm{Ker}(T)$, of a linear transformation T. Show that $\mathrm{Ker}(T)$ is a subspace of V.

(b) Let $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by $T(\mathbf{x}) = A \cdot \mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^5$, where A is the the folowing matrix : $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$.

(i) Let a, b be two fixed real numbers. Find the inverse image of the vector $(a, b, 0, 2a)$; that is, find all vectors $(x, y, z, t, u) \in \mathbb{R}^5$ such that $T(x, y, z, t, u) = (a, b, 0, 2a)$.

(ii) Consider the set of vectors found in (i). Is it a subspace of $\mathbb{R}^5$?

(iii) Find a basis of $\mathrm{Im}(T)$. What is the dimension of $\mathrm{Ker}(T)$?

3. **(a)** Define what it means for two $n \times n$ matrices A and B with coefficients in a field K to be *similar*. Show that similar matrices have the same determinant.

(b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation described by the following matrix

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

relative to the standard basis of $\mathbb{R}^3$. Find the characteristic polynomial of T, its eigenvalues and the corresponding eigenvectors. Is T diagonalisable ? If not find an invertible matrix S such that $S^{-1}BS = U$ is a triangular matrix. Write down the matrix U.

4. **(a)** Let V be an inner product space. Define what it means for a list of vectors $\{v_1, v_2, \dots, v_k\}$ in V to be an *orthogonal* set. Show that if $S = \{v_1, v_2, \dots, v_k\}$ is an orthogonal set of *non-zero* vectors in V, then S is a linearly independent set.

(b) Use the Gram-Schmidt orthogonalisation process to find an ortho*normal* basis for the vector subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = (2, -2, 0, 0), \quad v_2 = (1, 5, 0, -3), \quad v_3 = (1, 4, 1, -1)\,.$$

(c) Use the Hamilton-Cayley theorem to show that the matrix

$$C = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

satisfies the relation $C^6 = 73C^2 + 72I_4$. Is C invertible ? If the answer is yes, then use the Hamilton-Cayley theorem to express C^{-1} as a linear combination of the matrices I_4, C, C^2, C^3.