UNIVERSITY OF ABERDEEN **SESSION 2006-07**

DEGREE EXAMINATION

MA2505 Probability

Monday 21 May 2007 (9 am to 11 am)

Answer **THREE** *questions. A list of power series and the table of the normal distribution are available.*

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

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1. In a factory, machine A manufactures 75% and machine B manufactures 25% of the total production of washers. Unfortunately, 3 out of every 1000 washers that machine A produces are defective while 7 out of 1000 washers that machine B produces are defective. A washer is chosen at random from the total production.

(a) Describe a probability space (Ω, P) suitable for this random experiment by listing all the 4 possible outcomes and their probabilities.

(b) What is the probability that the washer chosen is defective? What is the probability that it was manufactured by machine A if it is found defective?

(c) The washers from this factory are packed in boxed of 1000 washers each. Use the central limit theorem to estimate the probability that a box contains at most 5 defective washers.

2. **(a)** Let $X_1, X_2, \dots$ and N be independent integer valued random variables. Assume that the X_n's have the same distribution as a random variable X. Let Z be the random variable defined by $Z = X_1 + \cdots + X_N$. Prove that the following equality of generating functions holds: $G_Z(t) = G_N(G_X(t))$. You may use without proof the fact that if $U \geq 0$ and V are independent random variable then $\int_{\{V=v\}} U = E(U) \cdot P(V = v)$ for any v.

(b) Show that $G_X(t) \leq 1$ for all $0 \leq t < 1$ and use Abel's lemma to prove that $\lim_{t \nearrow 1} G_X(t) = 1$. By applying the chain rule to $G_Z(t)$, prove that $E(Z) = E(N) \cdot E(X)$.

3. The probability that a family of Martians has k children where $k \geq 1$ is $(\frac{2}{5})^k$.

(a) What is the probability that a family of Martians has no children?

(b) Let X be the random variable describing the number of children in a family of Martians. Show that the generating function of X is $G_X(t) = \frac{1}{3} + \frac{2t}{5-2t}$. Find $E(X)$ and $\mathrm{Var}(X)$. You may use without proof the fact that $E(X^2) = \lim_{t \nearrow 1} [G''_X(t) + G'_X(t)]$.

(c) Assume that boys and girls appear in equal probability in Martian families independently of their size. Show that the probability that a family of Martians has *exactly* $n \geq 1$ boys is $\frac{5}{4^{n+1}}$.

4. A fair die has one face showing 1, two faces showing 2 and three faces showing 3. Players A and B roll the die. If the die shows 1, player A scores 4 points while player B scores none. If the die shows 2 then player B scores 3 points and player A scores none. If the die shows 3 both players score 1 point.

(a) Let X be the random variable of the number of points that player A scores, and let Y be the random variable of the number of points that player B scores. Calculate the expected values and variances of X and Y. Show that the random variable XY has the Bernoulli distribution with parameter 0.5. Show that $\text{Cov}(X, Y) = -\frac{5}{4}$.

(b) Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be random variables. Assume that all the X_i's are independent and that all the Y_i's are independent. Assume further that X_i and Y_j are independent if $i \neq j$. Set $U = X_1 + \cdots + X_n$ and $V = Y_1 + \cdots + Y_n$. Show that $\text{Cov}(U, V) = \sum_{i=1}^{n} \text{Cov}(X_i, Y_i)$. You may assume that all the values $E(X_i), E(Y_i)$ and $E(X_i Y_j)$ exists and are finite.

(c) Assume that players A and B play a sequence of n independent games. Let U denote the score of player A and let V denote the score of player B. Use parts (a) and (b) to show that $\text{Cov}(U, V) = -\frac{5}{4}n$.