DEGREE EXAMINATION

MA2504 Linear Algebra

Tuesday 30 May 2006 (noon—2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define the concept of a *basis* of a vector space V over a field K.

(b) Find bases for the row space $\mathrm{Row}(A)$, column space $\mathrm{Col}(A)$, and null space $N(A)$ of the real matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 3 & 2 \\ 2 & 1 & 2 & 5 & 3 \\ 2 & 0 & 1 & -3 & 2 \end{bmatrix} .$$

(c) Consider the following ordered set of vectors in the real vector space $\mathbb{R}^3$

$$\mathcal{F} = (\ f_1 = (1,0,-5), \quad f_2 = (0,1,-1), \quad f_3 = (-1,1,5)\ ) .$$

(i) Show that $\mathcal{F}$ is a basis for $\mathbb{R}^3$.

(ii) Find the transition matrix $[Id]^{\mathcal{E}}_{\mathcal{F}}$ from the standard ordered basis $\mathcal{E}$ of $\mathbb{R}^3$ to $\mathcal{F}$. Recall that $\mathcal{E} = (\ e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\ )$.

2. **(a)** Define the concept of a *linear transformation* $T\colon V \to W$, where V and W are vector spaces over a field K. Define the *kernel*, $\mathrm{Ker}(T)$, of a linear transformation T. Show that a linear transformation $T\colon V \to W$ is one-to-one if and only if its kernel consists only of the zero-vector 0_V.

(b) Let $T\colon \mathbb{R}^4 \to \mathbb{R}^5$ be the linear transformation defined by $T(\mathbf{x}) = A \cdot \mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^4$, where A is the the folowing matrix :

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 4 & 4 \\ 2 & 3 & 7 & 4 \end{bmatrix} .$$

(i) Let a, b be two fixed real numbers. Find the inverse image of the vector $(0, a, 3, a, b)$; that is, find all vectors $(x, y, z, t) \in \mathbb{R}^4$ such that $T(x, y, z, t) = (0, a, 3, a, b)$.

(ii) Consider the set of vectors found in (i). Is it a subspace of $\mathbb{R}^4$?

(iii) Find a basis of $\mathrm{Im}(T)$. What is the dimension of $\mathrm{Ker}(T)$?

3. **(a)** Let $T\colon V \to V$ be a linear transformation, where V is a vector space over a field K. Define what it means for a scalar $\lambda \in K$ to be an *eigenvalue* of T and for a vector $v \in V$ to be an *eigenvector* of T.

(b) Let $T\colon V \to V$ be a linear transformation and let $\lambda \in K$ be an eigenvalue of T. Show that λ^2 is an eigenvalue of the linear transformation $T \circ T$.

(c) Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation described by the following matrix

$$B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

relative to the standard basis of $\mathbb{R}^4$. Find the characteristic polynomial of T, its eigenvalues and the corresponding eigenvectors. Find an invertible matrix U such that $U^{-1}BU = D$ is a diagonal matrix. Write down the matrix D.

4. **(a)** Define what it means for two $n \times n$ matrices A and B with coefficients in a field K to be *similar*. Show that similar matrices have the same characteristic polynomial, the same eigenvalues and the same determinant.

(b) Use the Gram-Schmidt orthogonalisation process to find an orthogonal basis for the vector subspace of $\mathbb{R}^5$ spanned by the vectors

$$v_1 = (1, -1, 0, 0, 0), \quad v_2 = (1, 5, 0, -3, 0), \quad v_3 = (2, 0, 0, 2, 3)\,.$$

(c) Use the Hamilton-Cayley theorem to show that the matrix

$$C = \begin{bmatrix} 3 & 0 & -5 & 0 \\ 4 & 1 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 7 & 0 & 2 & -3 \end{bmatrix}$$

satisfies the relation $C^6 = 91C^2 - 90I$. Is C invertible ? If the answer is yes, then use the Hamilton-Cayley theorem to express C^{-1} as a linear combination of the matrices I, C, C^2, C^3.