**UNIVERSITY OF ABERDEEN** **SESSION 2004-05**

DEGREE EXAMINATION
MA2504 Linear Algebra
Tuesday 24 May 2005 (12 noon—2 pm)

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*Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.*

*Marks may be deducted for answers that do not show clearly how the solution is reached.*

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*Answer THREE questions. All questions carry equal weight.*

**1.** **(a)** Define the concept of a vector subspace of a vector space $V$ over a field $\mathbb{F}$.

**(b)** Find bases for the row space $\mathrm{Row}(A)$, column space $\mathrm{Col}(A)$ and null space $\mathrm{Nul}(A)$ of the real valued matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

**(c)** Show that the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible, determine $B^{-1}$ and express $B^{-1}$ as a product of elementary matrices.

**(d)** Let $V$, $W$ be vector spaces over a field $\mathbb{F}$ and let $T\colon V \to W$ be a linear transformation. Assume that $\dim(V) \gneqq \dim(W)$. Show that $T$ cannot be 1–1.

**2.** **(a)** Let $V$ and $W$ be vector spaces over a field $\mathbb{F}$. Define what it means for a linear transformation $T\colon V \to W$ to be an *isomorphism*.

**(b)** Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 3 & 0 & 1 & 1 \\ 1 & 2 & 1 & 5 \end{bmatrix}.$$

Determine whether $T$ is one–one and also whether it is onto.

**(c)** Compute the determinant of the matrix

$$B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & -1 & -5 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 1 & 5 & 4 \end{bmatrix}.$$

Use the result to determine whether $B$ is invertible.

**(d)** The multiplicative order of an invertible matrix $A$ is the least positive integer $n$ such that $A^n = I$. Define what it means for two matrices $A$ and $B$ to be similar. Show that similar matrices have the same order.

**3.** **(a)** Find the characteristic polynomial and the eigenvalues of the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix}.$$

**(b)** Find eigenvectors corresponding to the eigenvalues you found in part **(a)**.

**(c)** Find an invertible matrix $U$ such that $U^{-1}BU = D$ is a diagonal matrix and write down the matrix $D$.

**(d)** Consider the following slight variation on the matrix $B$ given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(The $(4, 4)$ entry is changed from $-2$ to 1.). Show that $C$ is not diagonalizable.

**4.** **(a)** Define what it means for an $n \times n$ matrix with real entries to be orthogonal. Show that a matrix $A$ is orthogonal if and only if $A^{-1} = A^T$.

**(b)** Use the Gram-Schmidt orthogonalisation process to find an orthogonal basis for the vector subspace of $\mathbb{R}^4$ spanned by the vectors $\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \}$.

**(c)** State the Cayley-Hamilton theorem. Let $A$ be a $4 \times 4$ matrix with real entries whose characteristic polynomial is $P_A(x) = x^4 - x^3 + x^2 - x + 1$. Use the Cayley-Hamilton theorem to show that $A^{10} = I$.