UNIVERSITY OF ABERDEEN **SESSION 2003–2004**

Degree Examination

MA2504 Linear Algebra

Friday 28 May 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Let V and W be vectors spaces over a field $\mathbb{F}$ and let $T\colon V \to W$ be a linear transformation. Define $\mathrm{Ker}(T)$, the *kernel of* T, and $\mathrm{Im}(T)$, the *image of* T. Show that $\mathrm{Ker}(T)$ is a vector subspace of V and that $\mathrm{Im}(T)$ is a vector subspace of W.

(b) Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 2 \\ 1 & 3 & -3 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 2 & 1 & 4 \end{pmatrix}.$$

Determine the dimensions of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$. Consider the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$. Write down the general solution of the system in parametric form.

2. **(a)** Let V and W be vector spaces over a field $\mathbb{F}$. Define what it means for a linear transformation $T\colon V \to W$ to be *1–1*, and what it means for it to be *onto*. Suppose V and W are vectors spaces over a field $\mathbb{F}$ of dimensions n and m respectively. Let $T : V \to W$ be a linear transformation. Show that if $n > m$ then T cannot be 1–1, and if $n < m$ then T cannot be onto.

(b) Calculate the determinant of the matrix

$$B = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -5 & -2 & 0 \\ 3 & 1 & -2 & 0 \\ 0 & 4 & -1 & 4 \end{pmatrix}.$$

Use the result to determine whether B is invertible.

(c) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix}.$$

Determine whether T is one–one and also whether it is onto.

3. **(a)** Let A be an $n \times n$ matrix with entries in a field $\mathbb{F}$. Define what it means for an element $\lambda \in \mathbb{F}$ to be an eigenvalue of A and for a vector $\mathbf{v} \in \mathbb{F}^n$ to be an eigenvector of A corresponding to the eigenvalue λ.

(b) Find the characteristic polynomial and the eigenvalues of the matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the eigenvectors corresponding to each eigenvalue. Find an invertible matrix U such that $U^{-1}BU = D$ is a diagonal matrix. Write down the matrix D.

(c) Consider the following slight variation on the matrix B given by

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(The $(4,1)$ entry is changed from 1 to 0.). Show that C is not diagonalisable.

4. **(a)** Define what it means for two $n \times n$ matrices A and B with entries in a field $\mathbb{F}$ to be similar. Show that similar matrices have the same determinant, the same characteristic polynomial and the same eigenvalues.

(b) Use the Gram-Schmidt orthogonalisation process to find an orthogonal basis for the vector subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(You may change the order of the vectors if you find it more convenient.)

(c) Use the Cayley-Hamilton theorem to show that the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 2 & -3 & -3 \\ 0 & 0 & -2 & 0 \\ 2 & 0 & -2 & -1 \end{bmatrix}$$

satisfies the relation $A^6 = 21A^2 - 20I$.