UNIVERSITY OF ABERDEEN **SESSION 2003–04**

Degree Examination
MA2504 Linear Algebra

Friday 13 August 2004 (3pm to 5pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define what it means for two $n \times m$ matrices A and B with entries in a field $\mathbb{F}$ to be row equivalent. Let $\bar{A} = [A|\mathbf{b}]$ and $\bar{C} = [C|\mathbf{d}]$ be augmented matrices corresponding to two systems of linear equations. Show that if $\bar{A}$ and $\bar{C}$ are row equivalent, then the corresponding systems have the same solution sets.

(b) Find bases for the null space and the column space of the real valued matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

(c) Determine whether the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is invertible, and if it is, calculate B^{-1}.

2. **(a)** Let V be a vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a set of vectors in V. Define what it means for $\mathcal{B}$ to be a basis for V, and what it means for V to be n-dimensional.

Show that if V is n-dimensional, then any set of n linearly independent vectors in V is a basis. You may assume that in an n-dimensional vector space any set of k vectors, where $k > n$ is linearly dependent.

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be the linear transformation whose standard matrix is

$$A_T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Determine if T is 1–1. Determine if T is onto. Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

(c) Let $\mathcal{A} = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$ and $\mathcal{B} = \{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$. It is given (you are not required to show) that $\mathcal{A}$ and $\mathcal{B}$ are bases for $\mathbb{R}^3$. Write down the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$.

3. **(a)** Let A be an $n \times n$ matrix with entries in a field $\mathbb{F}$. Define what it means for an element $\lambda \in \mathbb{F}$ to be an eigenvalue of A and for a vector $\mathbf{v} \in \mathbb{F}^n$ to be an eigenvector of A corresponding to an eigenvalue λ. Show that if A and B are similar matrices, then A and B have the same characteristic polynomial and hence the same eigenvalues.

(b) Find the characteristic polynomial and the eigenvalues of the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Find all eigenvectors corresponding to each eigenvalue of B. Also, find an invertible matrix U and a diagonal matrix D, such that $U^{-1}BU = D$.

(c) Consider the following matrix

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Explain why C is not diagonalisable.

4. **(a)** Define what it means for an $n \times n$ matrix with real entries to be orthogonal. Show that a matrix A is orthogonal if and only if $A^{-1} = A^T$.

(b) Use the Gram-Schmidt orthogonalisation process to find an orthogonal basis for the vector subspace of $\mathbb{R}^4$ spanned by the vectors $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}$.

(c) State the Cayley-Hamilton theorem. Let A be a 4×4 matrix with real entries whose characteristic polynomial is $P_A(x) = x^4 - x^3 + x^2 - x + 1$. Use the Cayley-Hamilton theorem to show that $A^{10} = I$.