UNIVERSITY OF ABERDEEN **SESSION 2005–06**

DEGREE EXAMINATION

MA2503 Introduction to Ordinary Differential Equations

Wednesday 24 May 2006 (3 pm—5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer all FOUR questions. All questions carry equal weight.

1. **(a)** Find the unique solution of the differential equation $x\dfrac{dy}{dx} - 2y = x^4$ for $x > 0$ that satisfies the initial condition, $y = 1$ when $x = 1$.

(b) Show that

$$y = \frac{2Ce^{x^2/2} - 1}{1 - Ce^{x^2/2}}$$

is a solution of the differential equation $\dfrac{dy}{dx} = x(y+1)(y+2)$, where C is any real constant. Is there any other solution?

(c) The population of a city was 2 millions in 1900 and had grown to 8 millions in 2000. Assuming the Malthusian Law of population growth, what will the population be in 2100? When will the population reach 10 millions?

2. **(a)** Let $\alpha_1(x)$ and $\alpha_2(x)$ be solutions of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \qquad (1)$$

for all real values of x. What additional condition must be satisfied for $\{\alpha_1(x), \alpha_2(x)\}$ to be a fundamental pair of solutions of (1)? When $\{\alpha_1(x), \alpha_2(x)\}$ is a fundamental pair of solutions, what is the general solution of (1)?

Let $\beta(x)$ be a solution of the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x) \qquad (2)$$

and $\alpha(x)$ a solution of (1). Prove that $y = \alpha(x) + \beta(x)$ is a solution of (2).

(b) Find the general solution of the differential equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 4y = 0$. Find also the solution satisfying the initial condition, $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.

(c) Use the method of undetermined coefficients to obtain a particular integral for the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x^2$. Hence give the general solution of this differential equation.

3. **(a)** Verify that $y = 1/x$ is a solution of the homogeneous equation associated with

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x, \qquad \text{for} \quad x > 0. \tag{3}$$

By setting $y = z/x$ in (3), find and solve a second order differential equation for z.

Give the general solution of (3) valid for all $x > 0$.

(b) Let

$$\mathbf{r} = (t^2 + t - 2)\mathbf{i} + (t + 2)\mathbf{j} + (t + 3)^2\mathbf{k} \qquad (t \in \mathbb{R})$$

be the vector equation of a curve in $\mathbb{R}^3$.

Show that this curve passes through the point A with position vector $\mathbf{k}$.

Find the vector and cartesian equations of the tangent line to the curve at a point P_0 corresponding to $t = t_0$.

Show that there is a unique point B on the curve at which the tangent is perpendicular to the tangent at A.

4. **(a)** A tank of capacity 1000 litres is half full of liquid A. Liquids A and B are pumped into the tank at a rate of 5 litres per minute each. The well-stirred mixture is drawn off at a rate of 5 litres per minute.

When is the tank full? At this time, what are the volumes of liquids A and B respectively in the tank?

(b) A particle P is projected with speed 5 metres per second at an angle of elevation $\pi/3$ over horizontal ground from a point O at ground level.

Find a vector equation for the position of the particle at time t before it hits the ground again.

What is the greatest vertical height that the particle reaches?

Find also the distance from O when it hits the ground. Give, without proof, another angle of elevation with the same speed of projection that would realize this same distance.