UNIVERSITY OF ABERDEEN **SESSION 2004–05**

Degree Examination

MA2503 Introduction to Ordinary Differential Equations

Friday 27 May 2005 (3 pm—5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer all FOUR questions. All questions carry equal weight.

1. **(a)** Find the general solution of the differential equation

$$x\frac{dy}{dx} + 2y = \frac{\sin x}{x}, \quad x > 0.$$

Find also the solution satisfying $y = 0$ when $x = \pi$.

(b) Find the general solution of the equation

$$2xy\frac{dy}{dx} = x^2 + 3y^2, \quad x > 0,$$

by substituting $y = xz$.

(c) The function $y = e^{2x}$ is a solution of the diferential equation $\frac{dy}{dx} = 2y$. By considering $\frac{d}{dx}\{e^{-2x}y\}$, show that every solution of $\frac{dy}{dx} = 2y$ is of the form $y = Ae^{2x}$ for some constant A.

2. **(a)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0.$$

Find also the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = e^{-2x}.$$

(b) Verify that $y = x$ is a solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Use the method of reduction of order to obtain the general solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3, \quad x > 0,$$

by setting $y = zx$.

Identify a pair of fundamental solutions of the homogeneous equation and verify that the Wronskian is non-zero.

3. **(a)** The population of a city in the year 2005 had increased by 10% from what it had been in the year 2000. Assuming the Malthusian Law of population growth, when will the population have increased by 25% compared with 2000?

(b) A large tank is partially filled with 10^3 litres of liquid A. A mixture of equal proportions by volume of liquid A and another liquid B is poured into the tank at a rate of 10 litres per second. The tank is well-stirred and the contents drawn off at a rate of 5 litres per second.

Find formulae for the volumes of liquid A and liquid B in the tank before the tank overflows.

If the tank has a capacity of 10^4 litres, how much liquid B is in the tank when it overflows?

4. **(a)** The position vectors of points on a curve are given by $\mathbf{r} = (t^3 + 3t)\mathbf{i} - t^4\mathbf{j} + (t^2 + 1)\mathbf{k}$. Show that the point A with position vector $-4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ lies on the curve and find the vector and coordinate equation of the tangent line to the curve at A.

Show also that this tangent line is perpendicular to the vector $\mathbf{i} + 3\mathbf{k}$ and that there is no point on the curve with tangent parallel to $\mathbf{i} + 3\mathbf{k}$.

(b) A constant force of 4 Newtons is applied to a particle of 3 kilograms in the direction of $4\mathbf{j} + 3\mathbf{k}$. Initially the particle is at the origin O and is moving with speed 2 metres per second in the direction of $\mathbf{i}$. Find the position vector of the particle t second later.

(c) A body is projected over horizontal ground at an angle of elevation α to the horizontal with an initial speed of u. Assuming that only the force of gravity acts on the body during flight, producing a constant acceleration of $-g\mathbf{k}$, find at what distance from its point of projection the body hits the ground.

What angle α should be chosen to maximize the distance?