## Degree Examination

MA2005 Introduction to Analysis
Monday 22 January 2007
(12 noon to 2 pm )

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FOUR questions. Each question has equal weight. The questions are independent of each other and can be treated in any order.

1. We define a sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ recursively as follows:

$$
a_{1}=1 \quad \text { and } \quad a_{n+1}=\frac{2 a_{n}+2}{a_{n}+2} \quad \text { for } n \geq 0
$$

Here are the first few values (you don't need to verify this):

$$
\begin{array}{c|ccccccc}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline a_{n} & 1 & \approx 1.3333 & \approx 1.4000 & \approx 1.4118 & \approx 1.4138 & \approx 1.4141 & \approx 1.4142
\end{array}
$$

It looks as if $\left(a_{n}\right)$ converges to $\sqrt{2}$, as $n \rightarrow \infty$. The purpose of this exercise is to prove this.
a) Show that $0<a_{n}^{2}<2$ for all $n$. (Use induction on $n$.)
b) Show that $a_{n+1}-a_{n}=\frac{2-a_{n}^{2}}{a_{n}+2}$ for all $n \in \mathbb{N}$.
c) State the "Principal of Monotone Sequences". Show that the sequence $\left(a_{n}\right)$ satisfies the conditions of this principle and deduce that $\left(a_{n}\right)$ converges. Determine $\lim _{n \rightarrow \infty}\left(a_{n}\right)$.
2. a) Let $h(x)=\sqrt{1+x^{2}}$ for $x \in \mathbb{R}$. Explain briefly why $h$ is differentiable and

$$
h^{\prime}(x)=\frac{x}{\sqrt{1+x^{2}}} \quad \text { for all } x \in \mathbb{R}
$$

State clearly which results of the course you are using. You may use without proof the fact that $\left(x^{\alpha}\right)^{\prime}=\alpha x^{\alpha-1}$ for any $\alpha$.
b) State Cauchy's Mean Value Property for two functions $f$ and $g$.
c) Let $x \in \mathbb{R}$. Show that there exists some $c$ between 0 and $x$ such that

$$
\sqrt{1+x^{2}}=1+\frac{1}{\sqrt{1+c^{2}}} \frac{1}{2} x^{2}
$$

Deduce that $\sqrt{1+x^{2}} \leq 1+\frac{1}{2} x^{2}$ for all $x \in \mathbb{R}$.
(Hint: If $x=0$, the assertion is clear. Now assume that $x>0$. In order to prove the existence of the required $c$, apply Cauchy's Mean Value Property to the functions $f(t)=\sqrt{1+t^{2}}-1$ and $g(t)=\frac{1}{2} t^{2}$, defined on the interval $[0, x]$.)
3. a) State
i) Rolle's Theorem;
ii) the Fundamental Theorem of Calculus (both version I and version II).
b) Let $f:[a, b] \rightarrow \mathbb{R}$ be a continuous function. Define a new function $F$ by

$$
F(x)=(b-a)\left(\int_{a}^{x} f(t) d t\right)-(x-a) \int_{a}^{b} f(t) d t \quad \text { for } x \in[a, b]
$$

Show that the assumptions of Rolle's Theorem are satisfied for $F$. Deduce that there exists some $c \in(a, b)$ such that

$$
f(c)=\frac{1}{b-a} \int_{a}^{b} f(t) d t
$$

4. a) Let $S$ be a non-empty set of real numbers. Define the terms "upper bound for $S$ " and "supremum of $S$ ".
b) Consider the following subset of $\mathbb{R}$ :

$$
S=\left\{\left.(-1)^{n}\left(1-\frac{1}{n}\right) \right\rvert\, n \in \mathbb{N}\right\}
$$

i) Write down the elements of $S$ corresponding to $n=1,2,3,4,5$.
ii) Show that $S$ is bounded above and below.
iii) Show that $1=\sup (S)$.
c) Let $a_{n}=(-1)^{n}\left(1-\frac{1}{n}\right)$ for $n \in \mathbb{N}$, that is, we have $S=\left\{a_{n} \mid n \in \mathbb{N}\right\}$. Show that the sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ is not convergent. (Hint. Consider the subsequences $\left(a_{2 k}\right)_{k \in \mathbb{N}}$ and $\left(a_{2 k-1}\right)_{k \in \mathbb{N}}$. Show that each of these subsequences has a limit.)
5. a) State the Intermediate Value Property for continuous functions.
b) Let $f:[0,1] \rightarrow \mathbb{R}$ be a continuous function such that $f(x) \in[0,1]$ for all $x \in[0,1]$. Show that there exists some $c \in[0,1]$ such that $f(c)=c$.
(Hint: Consider the function $g(x)=f(x)-x$ for $x \in[0,1]$. Show that $g(0) \geq 0$ and $g(1) \leq 0$. Now apply the Intermediate Value Property and deduce that $g(c)=0$ for some $c \in[0,1]$.)
c) Let $f(x)=x^{5} \exp (x)+1$ for $x \in \mathbb{R}$. Show that there exists some $n \in \mathbb{N}$ such that $f(n)>0$ and $f(-n)<0$. Deduce that there exists some $c \in(-n, n)$ such that $f(c)=0$.

