UNIVERSITY OF ABERDEEN **SESSION 2005–06**

DEGREE EXAMINATION

MA2003 Advanced Calculus

Friday 27 January 2006 (3 pm to 5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FOUR questions. (All questions carry equal weight.)

Please note that full explanations form an essential part of your solution.

1. Graph sketching.

Consider the function defined by the formula $f(x) = \dfrac{e^x}{x^3 - 1}$.

(a) (i) Find the domain of definition of f.

(ii) What happens to $f(x)$ as x tends to $\pm\infty$ or to a point where $f(x)$ is not defined?

(b) Show that $f'(x)$ has the form $f'(x) = \dfrac{e^x \cdot p(x)}{(x^3 - 1)^2}$, and find $p(x)$.

(c) Find an integer n such that the equation $p(x) = 0$ has a solution in the interval $[n, n + 1)$. (Which results of the course are you using here?)

(d) Show that the equation $p(x) = 0$ has exactly one solution. (*Hint:* Use $p'(x)$ to find the regions where $p(x)$ is increasing or decreasing.)

(e) (i) Find the regions where $f(x)$ is increasing or decreasing. (If you haven't done (d), you may use here without proof the fact that $p(x) = 0$ has only one solution.)

(ii) Sketch the graph of $f(x)$ for $-2 \le x \le 10$.

(iii) What is the range of f?

2. Limits and Taylor's Theorem.

Let $\alpha \in \mathbb{R}$, $\alpha > 0$, be fixed. Consider the function $f\colon (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = x^\alpha (e^{1/x} - 1) \qquad \text{for } x > 0.$$

(a) (i) Assume that $0 < \alpha \le 1$. Then show that $\lim_{x\to\infty} f(x)$ exists, and compute the limit. To do this, it may be useful to write $f(x) = g(x)/h(x)$ for suitable $g(x)$ and $h(x)$ and use l'Hôpital's rule.

(ii) Assume that $\alpha > 1$. Show that the limit does not exist. (*Hint:* First try $\alpha = 2$; set $y = 1/x$ and use the Taylor formula $e^y = 1 + y + \frac{1}{2}y^2 e^c$ where $0 < c < y$.)

(b) Let $\alpha = 1$, so that $f(x) = x(e^{1/x} - 1)$.

(i) Find the Taylor polynomial $P_2(x)$ of degree two for $f(x)$ about $x = 1$. Recall that

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2} f''(1)(x - 1)^2.$$

(ii) Show that if $|x - 1| < 0.1$ then the error in approximating $f(x)$ by $P_2(x)$ is less than 0.004.

3. PDE.

Let D be the set of all $(x, y) \in \mathbb{R}^2$ where $x > 0$. Consider functions $f\colon D \to \mathbb{R}$ satisfying the PDE (partial differential equation)

$$x\,\frac{\partial f}{\partial x}(x,y) - y\,\frac{\partial f}{\partial y}(x,y) = xy^2 \qquad \text{for all } (x,y) \in D. \tag{$*$}$$

(a) Consider a change of variables such that

$$x = x(s,t) = s \qquad \text{and} \qquad y = y(s,t) = t/s, \qquad \text{where } (s,t) \in D.$$

Set $F(s,t) = f(x(s,t), y(s,t)) = f(s, t/s)$. Express $\dfrac{\partial F}{\partial s}$ and $\dfrac{\partial F}{\partial t}$ in terms of $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and the partial derivatives of x, y with respect to s, t.

(b) Show that

$$\frac{\partial F}{\partial s}(s,t) = t^2/s^2 \qquad \text{for all } (s,t) \in D;$$

find all functions $F(s,t)$ satisfying this condition.

(c) Find all functions $f(x,y)$ satisfying the PDE $(*)$.

4. Minima/Maxima.

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = y(x^2 - e^{-y}).$$

(a) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, and determine the critical points of f.

(b) Compute $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$. Determine at which of the critical points f has a local minimum or maximum.

(c) Does f have a global minimum or maximum?

5. Multiple integrals.

(a) Let $E \subseteq \mathbb{R}^2$ be a non-empty subset such that its area, denoted $\mu(E)$, is finite. As in the course, set

$$x_0 = \frac{1}{\mu(E)} \int\!\!\int_E x\,dxdy \qquad \text{and} \qquad y_0 = \frac{1}{\mu(E)} \int\!\!\int_E y\,dxdy,$$

and call the point $(x_0, y_0) \in \mathbb{R}^2$ the *centre of gravity* of E. Determine the centre of gravity for:

$$E = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } \frac{x}{a} + \frac{y}{b} \leq 1\}, \quad \text{where} \quad a, b > 0.$$

(b) Let $E := \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0, x^2 + y^2 - 2y \geq 0, x^2 + y^2 \leq 1\}$.

(i) Sketch the region E.

(ii) Let $f(x,y)$ be a function. Use polar coordinates to show that

$$\int_E f(x,y)\,dxdy = \int_F r f(r\cos(\theta), r\sin(\theta))\,drd\theta,$$

where $F = \{(r,\theta) \mid 0 \leq \theta \leq \pi/6 \text{ and } 2\sin(\theta) \leq r \leq 1\}$.

(iii) Compute $\int_E dxdy$, the area of E.