UNIVERSITY OF ABERDEEN **SESSION 2004–05**

Degree Examination

MA2003 Advanced Calculus

Thursday 20 January 2005 (12noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FOUR questions. All questions carry equal weight.

1. **(a)** Let $f(x) = 12 - 6x - 3x^2 - x^3$. Find $f'(x)$ and $f''(x)$. Show that f has no critical points but has one point of inflexion.

Find the integer n such that f has a root in the open interval $(n, n+1)$. Justify your answer.

Explain why f has no other roots.

(b) Let $g(x, y) = x^4 + y^4 - 4xy$. Find and classify the three critical points of g.

2. **(a)** Let $f(x, y) = y \cos\left(\dfrac{x}{y}\right)$ where $y \neq 0$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Verify that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

(b) A cylinder has height h cm and a circular base with radius r cm. Both h and r are varying with time. At a certain instant, the height is 2 metres and is increasing at 2 cm/second, and the radius is 1 metre and is decreasing at 1 cm/second. What is the rate of change of the volume at this instant? You may leave your answer in terms of π.

(c) Suppose that $z = g(x, y)$ and let $u = \dfrac{x}{y}$ and $v = xy$ (for $x, y > 0$). Show that

$$\frac{\partial z}{\partial x} = \frac{1}{y}\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}$$

and obtain a similar expression for $\dfrac{\partial z}{\partial y}$.

Hence show that the general solution of the partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \qquad (x, y > 0)$$

has the form $z = h\left(\dfrac{x}{y}\right)$ where h is an arbitrary differentiable function of one variable. Suppose further that $g(x, 1) = x^2$ for all $x > 0$. Find z in this case.

3. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$. Define what is meant by saying that f is (i) *increasing* on $\mathbb{R}$, (ii) *strictly increasing* on $\mathbb{R}$.

Let $a, b \in \mathbb{R}$. Verify that

$$(b-a)\left((b+\frac{a}{2})^2 + \frac{3a^2}{4}\right) = b^3 - a^3.$$

Hence, or otherwise, show that if $f(x) = x^3$ for all $x \in \mathbb{R}$ then f is strictly increasing on $\mathbb{R}$.

(b) For each of the following statements (i), (ii) and (iii), write down whether it is true or false. If the statement is false, give a counter-example. If the statement is true, give a proof. [You may assume that a limit of non-negative Newton quotients is non-negative. You may also use the Mean Value Theorem without proof, provided that you check that the hypotheses of that theorem are satisfied in the situation in which you wish to apply it.]

(i) If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on $\mathbb{R}$ then $f'(x) > 0$ for all $x \in \mathbb{R}$.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$ then f is strictly increasing on $\mathbb{R}$.

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing on $\mathbb{R}$ then $f'(x) \geq 0$ for all $x \in \mathbb{R}$.

4. **(a)** Determine the following limits, giving reasons for your answers:

$$\lim_{x\to 0} \frac{x^2+x-2}{x^2-1}; \qquad \lim_{x\to 1} \frac{x^2+x-2}{x^2-1}; \qquad \lim_{x\to \infty} \frac{6+x-2x^2}{7-x+4x^2}; \qquad \lim_{x\to 0} \frac{x^2}{\sin(4x)}.$$

(b) State Rolle's Theorem. Name (but do not state) another important theorem which can be proved by using Rolle's Theorem.

(c) Let $f(x) = \cos(2x)$. Find the Taylor polynomial $P_4(x)$ of degree 4 for f about $x = 0$. Show that if $|x| < 0.1$ then the error in approximating $f(x)$ by $P_4(x)$ is less than $\frac{4}{15} \times 10^{-5}$.

5. **(a)** Find the volume of the region below the surface $z = x\sin(\pi y)$ and above the rectangle

$$R = \{(x, y) : 0 \leq x \leq \pi,\ 0 \leq y \leq 1\}.$$

(b) Evaluate $\iint_T x^2 y\, dx\, dy$ where T is the triangle bounded by the x and y axes and the line $x + y = 3$.

(c) Use polar coordinates to evaluate $\iint_R e^{x^2+y^2}\, dx\, dy$ where R is the region lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. [You may leave your answer in terms of π and e.]