UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION

MA2003 Advanced Calculus

Tuesday 20 January 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FOUR questions. All questions carry equal weight.

1. **(a)** Let $f(x) = x^3 + 3x^2 + 6x - 15$. Find $f'(x)$ and $f''(x)$.

Show that f has no critical points but has one point of inflexion.

Evaluate f at the point of inflexion.

Prove that f has *exactly* one real root and find the integer n such that the root lies in the open interval $(n,\ n+1)$.

(b) Let $f(x, y) = 3x^2 - 6xy + 6y^2 - 2y^3$. Find and classify the critical points of f.

2. **(a)** Let $f(x, y) = x \ln(xy)$ for $x, y > 0$. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and verify that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

(b) Let $f(x, y) = 3x^2 - 2xy + x^2 y$. Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(1, 1, 2)$.

(c) Suppose that $z = f(x, y)$ and let $s = \dfrac{x}{y}$ and $t = xy$ (for $x, y > 0$). Show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{1}{y} + \frac{\partial z}{\partial t}\, y$$

and obtain a similar expression for $\dfrac{\partial z}{\partial y}$.

Hence show that this change of variables reduces the partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4y^2$$

to the form

$$\frac{\partial z}{\partial t} = \frac{2}{s}.$$

Deduce that $z = 2y^2 + g\left(\dfrac{x}{y}\right)$ where g is an arbitrary differentiable function of one variable.

Suppose further that $f(x, 1) = x^2$ for all $x > 0$. Find z in this case.

3. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$. Define precisely what is meant by saying that f is *strictly increasing on* $\mathbb{R}$.

Prove that, if f is strictly increasing on $\mathbb{R}$ and g is strictly decreasing on $\mathbb{R}$, then the composition $f \circ g$ is strictly decreasing on $\mathbb{R}$.

(b) Let f be a function which is continuous on $[1, \infty)$ and suppose that $f'(x) > 0$ for all $x > 1$. Use the Mean Value Theorem to prove that f is strictly increasing on $[1, \infty)$. By considering $x^5 - 1 - 5(x - 1)$, or otherwise, deduce that

$$x^5 - 1 > 5(x - 1)$$

for all $x > 1$.

(c) Let

$$f(x) = xg(x)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $g(1) = 0$. Calculate $f'(x)$. Explain why Rolle's Theorem can be applied to f on the interval [0,1]. Deduce that there exists $c \in (0, 1)$ such that

$$g'(c) = -\frac{g(c)}{c}.$$

4. **(a)** Determine the following limits and explain how you obtain your answers:

$$\lim_{x\to 2} \frac{2x^2 - x - 6}{x^2 - 4}, \qquad \lim_{x\to 1} \frac{2x^2 - x - 6}{x^2 - 4},$$

$$\lim_{x\to \infty} \frac{2x^2 - x - 6}{x^2 - 4}, \qquad \lim_{x\to \infty} \frac{\ln x}{x^{1/4}}.$$

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the point a. Express $f(x)$ in terms of the Newton quotient $\dfrac{f(x) - f(a)}{x - a}$. [Hint. $f(x) = (f(x) - f(a)) + f(a)$.]

By using the "algebra of limits", or otherwise, deduce that f is continuous at the point a.

(c) Let $f(x) = \sin(2x)$. Find the Taylor polynomial $P_3(x)$ of degree 3 for f about $x = 0$.

Show that if $|x| < 0.1$ then the error in approximating $f(x)$ by $P_3(x)$ is less than 0.67×10^{-4}.

5. **(a)** Find the volume of the region under the surface $z = 4xy + y^2$ and above the rectangle

$$R = \{(x, y) : 1 \le x \le 2,\ 0 \le y \le 1\}.$$

(b) Sketch the region of integration for

$$\int_{x=0}^{\infty} \int_{y=-\infty}^{\infty} e^{-\sqrt{x^2+y^2}}\, dy\, dx.$$

Evaluate the integral by using polar coordinates.

(c) Evaluate $\displaystyle\iint_T xy\, dx\, dy$ where T is the triangle in the first quadrant bounded by the x and y axes and the line $x + y = 1$.