UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION

MA2003 Advanced Calculus

Tuesday 10 August 2004 (12noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FOUR questions. All questions carry equal weight.

1. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3 + 6x^2 - 2x^3$.

Find the critical points of f. Find the intervals on which f is strictly increasing and the intervals on which f is strictly decreasing.

Hence, or otherwise, classify the critical points of f.

Prove that the equation $f(x) = 0$ has a unique solution x_0 and find the integer n such that $n < x_0 < n + 1$.

Show that f has a point of inflexion.

Determine the behaviour of $f(x)$ as $x \to \infty$ and the behaviour of $f(x)$ as $x \to -\infty$.

Using the information that you have obtained, sketch the graph of f.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 3x + 2x^3 - \frac{1}{2}x^4$.

How is the function f of part (a) related to g?

Assuming the fact that the equation $f(x) = 0$ in part (a) has a unique solution, prove that the equation $g(x) = 10$ cannot have three distinct solutions.

[Hint. Suppose that there are three distinct solutions and apply Rolle's Theorem to g to obtain a contradiction.]

2. **(a)** Let $f(x, y) = x \sin\left(\dfrac{y}{x}\right)$, where $x \neq 0$.

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Verify that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

(b) Let $f(x, y) = xy^2 - 2xy + 3y^2$. Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(1, 1, 2)$.

(c) Suppose that $z = f(x, y)$, and let $s = xy$ and $t = \dfrac{x}{y}$ (for $x, y > 0$). Show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s} y + \frac{\partial z}{\partial t} \frac{1}{y}$$

and obtain a similar expression for $\dfrac{\partial z}{\partial y}$.

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Hence show that this change of variables reduces the partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2$$

to the form

$$\frac{\partial z}{\partial s} = t.$$

Deduce that $z = x^2 + g\left(\frac{x}{y}\right)$ where g is an arbitrary differentiable function of one variable.

Suppose further that $f(x, 1) = 0$ for all $x > 0$. Find z in this case.

3. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$. Define precisely what is meant by saying that f is *strictly decreasing on* $\mathbb{R}$.

Prove that if f is strictly decreasing on $\mathbb{R}$ then the composition $f \circ f$ is strictly increasing on $\mathbb{R}$.

In the special case in which $f(x) = -x^3$, what is the value of $(f \circ f)(x)$?

Suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfies $g'(x) < 0$ for all x. Use the Mean Value Theorem to prove that g is strictly decreasing on $\mathbb{R}$.

(b) Let $f(x, y) = 2x^3 - 6x^2 + 6xy - 3y^2$. Find and classify the critical points of f.

4. **(a)** Determine the following limits and explain how you obtain your answers:

$$\lim_{x\to\infty} \frac{2x^3 + x - 4}{x^3 - x^2 + 6}, \qquad \lim_{x\to\pi/2} \frac{\cos x}{x - \pi/2},$$

$$\lim_{x\to 0} \frac{\cos x}{x - \pi/2}, \qquad \lim_{x\to\infty} \frac{\ln x}{x^{1/2}}.$$

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the point a. Express $f(x)$ in terms of the Newton quotient $\dfrac{f(x) - f(a)}{x - a}$. [Hint. $f(x) = (f(x) - f(a)) + f(a)$.]

By using the "algebra of limits", or otherwise, deduce that f is continuous at the point a.

(c) Let $f(x) = \ln(1 + x)$ for $x > -1$. Find the Taylor polynomial $P_2(x)$ of degree two for f about $x = 0$.

Show that if $0 < x < 0.1$ then the error in approximating $f(x)$ by $P_2(x)$ is less than $\frac{1}{3} \times 10^{-3}$.

5. **(a)** Find the volume of the region under the surface $z = xy + 3xy^2$ and above the rectangle

$$R = \{(x, y) : 1 \le x \le 2,\ 0 \le y \le 1\}.$$

(b) Let D be the semi-circular region of radius a (> 0) given by

$$x^2 + y^2 \leqslant a^2, \qquad y \geq 0.$$

By using polar coordinates, or otherwise, evaluate

$$\iint_D y \; dx \, dy.$$

Hence show that the y-coordinate of the centroid of D is $\dfrac{4a}{3\pi}$.

(c) Sketch the region of integration for

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \; dx \; dy.$$

Change the order of integration, and hence evaluate the integral.