UNIVERSITY OF ABERDEEN **SESSION 2005–06**

DEGREE EXAMINATION
MA1502 Algebra
Friday 26 May 2006 (noon—2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer all SIX questions in section A and TWO questions from section B.

Each question in section A is worth 10 marks and each question in section B is worth 20 marks. Justify your answers. Answers without justification earn fewer marks and in some cases no marks at all.

SECTION A

1. **(a)** Determine the real and imaginary parts of the complex number

$$\frac{2+5i}{1-4i}.$$

(b) Sketch the following four complex numbers on an Argand diagram (*i.e.* depict them as "points in the plane"). For each of the four complex numbers, determine the modulus and the principal argument:

$$-1+i\ ,\quad 2-i\sqrt{12}\ ,\quad \cos(\pi/7)+i\sin(\pi/7)\ ,\quad \sin(\pi/7)+i\cos(\pi/7).$$

2. Find the real and imaginary parts of $(-2+2i)^{20}(1+i\sqrt{3})^{12}$.

3. Three points in 3-space are given by $A=(2,1,0)$, $B=(5,1,-1)$ and $C=(-2,0,2)$. Calculate the following:

(i) $\underline{AB}\cdot\underline{AC}$,

(ii) the cosine of the angle at A of the triangle ABC,

(iii) the area of the triangle ABC,

(iv) the value of $\lambda\in\mathbb{R}$ such that $\underline{AB}+\lambda\underline{BC}$ is perpendicular to $\underline{BC}$.

4. **(a)** Let $A=\begin{bmatrix}1&0&0\\2&0&2\end{bmatrix}$ and $B=\begin{bmatrix}2&3\\2&1\\-3&-1\end{bmatrix}$. Calculate AB.

(b) Let $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Determine all 2×2 matrices B such that $AB-BA=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

5. Let

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Use mathematical induction to show that the following is true for every integer $n \geqslant 1$:

$$C^n + C^{n+1} = \begin{bmatrix} 2 \cdot 4^n & 2 \cdot 4^n \\ 3 \cdot 4^n & 3 \cdot 4^n \end{bmatrix}.$$

6. **(a)** Evaluate $\binom{7}{4}$ and $\binom{15}{11}$.

(b) A standard pack of cards contains $52 = 13 \times 4$ cards (13 ranks in four suits: the ranks are in increasing order 2,3,4,. . . ,10, Jack, Queen, King, Ace and the suits are clubs, spades, hearts and diamonds)

(i) Three cards are dealt without replacement from a well-shuffled pack. What is the probability that none of them is an Ace ?

(ii) Four cards are dealt without replacement from a well-shuffled pack. What is the probability that no two of them belong to the same suit ?

(iii) 20 cards are dealt without replacement from a well-shuffled pack. What is the probability that at least one of them is a Queen ?

SECTION B

7. **(a)** Show that $-1 + i$ is a solution of the equation

$$x^6 + 2x^5 + 3x^4 + 2x^3 + 3x^2 + 2x + 2 = 0$$

and find all the other solutions (real or complex) of this equation.

(b) Find all solutions of the equation $z^5 = 4-4i$. For each solution, write down the modulus and principal argument.

8. **(a)** Let M be the plane with equation $x + 4y - 5z = 1$.
Write down a nonzero vector which is perpendicular to the plane.

Let A and B be the points given by $A = (0, 3, 5)$ and $B = (-1, 2, 4)$. Find the vector equation of the straight line L through A and B. Decide whether L is parallel to M ; if not, determine the coordinates of the point P where L meets M.

Calculate the distance from A to M.

(b) By using row reduction, find all solutions of the simultaneous equations

$$\begin{array}{rrcrcrcr} w & +x & - & 2y & + & z & = & 1 \\ 2w & +2x & & & + & 2z & = & -1 \\ & x & + & 6y & + & 4z & = & 3\,. \end{array}$$

9. **(a)** Suppose that a probabilistic experiment has finitely many distinct possible outcomes $s_1, s_2, \ldots, s_n$ which occur with probabilities $p(s_1), p(s_2), \ldots, p(s_n)$ respectively. Define what is meant by an *event* associated with the experiment. Define what is meant by the *probability* $P(A)$ of an event A. Define what is meant by the conditional probability $P(A|B)$ for events A and B.

(b) Three fair dice coloured yellow, red and blue are rolled at the same time (each die has faces numbered 1 to 6).

Let A be the event that the score of the yellow die is divisible by 3.
Let B be the event that the red and the blue die have distinct scores.
Let C be the event that the sum of the scores on the three dice equals 10.

Determine the following probabilities:

(i) $P(A)$

(ii) $P(B)$

(iii) $P(C)$

(iv) $P(A \cap B)$

(v) $P(A|B)$

(vi) $P(A|C)$.

Are A and B independent events?
Are A and C independent events?
Are B and C independent events?