Degree Examination
MA1502 Algebra
Wednesday 1 June 2005 (12 noon—2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer all SIX questions in section A and TWO questions from section B.

Each question in section A is worth 10 marks and each question in section B is worth 20 marks.

SECTION A

1. **(a)** Find the real and imaginary parts and the complex conjugate of the complex number

$$\frac{1-4i}{3+2i}.$$

(b) Find the modulus and the principal argument of each of the complex numbers

$$-1-i\ , \qquad 3+i\sqrt{3}\ , \qquad 10(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12})\,.$$

2. Find the real and imaginary parts of $(1+i)^{50}(1-i\sqrt{3})^{18}$.

3. Let A, B and C be points in 3–dimensional space with position vectors

$$\mathbf{a}=\begin{bmatrix}3\\2\\0\end{bmatrix}, \qquad \mathbf{b}=\begin{bmatrix}1\\0\\-2\end{bmatrix}, \qquad \mathbf{c}=\begin{bmatrix}0\\-1\\3\end{bmatrix},$$

respectively. Calculate the following:

(i) $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})$;

(ii) the cosine of the angle at C of the triangle ABC;

(iii) $\mathbf{a}\times\mathbf{b}$;

(iv) the area of the triangle OAB;

(v) the value of λ such that $\mathbf{c}+\lambda(\mathbf{b}-\mathbf{c})$ is perpendicular to $\mathbf{c}$.

4. **(a)** Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Find all matrices $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $AB = BA$.

(b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 4 & -1 \end{bmatrix}$. Find $\det(A)$ and $\det(A^4)$.

5. **(a)** Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$. Write down the transpose A^T of A . Calculate AA^T and A^TA.

(b) Let $C = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Find C^{-1} and C^2.

Use mathematical induction to show that $C^n = \begin{bmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{bmatrix}$ for all $n \geqslant 1$.

6. **(a)** Evaluate $\binom{11}{5}$ and $\binom{8}{4}$.

(b) A standard pack of cards contains $52 = 13 \times 4$ cards [13 ranks in four suits: the ranks are in increasing order 2,3,4,. . . ,10, Jack, Queen, King, Ace and the suits are clubs, spades, hearts and diamonds.]

(i) Three cards are dealt without replacement from a well–shuffled pack. What is the probability that none of them is a King ?

(ii) Four cards are dealt without replacement from a well–shuffled pack. What is the probability that at least one of them is a Queen ?

(iii) Three cards are dealt without replacement from a well–shuffled pack. What is the probability that they are of three distinct ranks ?

SECTION B

7. **(a)** Show that $2 + i$ is a solution of the equation

$$x^4 - 4x^3 + 7x^2 - 8x + 10 = 0$$

and find all the other solutions [real or complex] of this equation.

(b) Find all solutions of the equation $z^5 = -2i$. For each solution, write down the modulus and principal argument.

8. **(a)** Let M be the plane with equation

$$3x - y + 2z = -6\,.$$

Write down a nonzero vector which is perpendicular to the plane.

Let A and B be the points with position vectors

$$\mathbf{a} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

respectively. Find the vector equation of the straight line L through A and B. Find the position vector of the point P where L meets M.

(b) By using row reduction, find all solutions of the simultaneous equations

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 2 \\ 2x & & & + & 2z & = & -1 \\ x & + & 6y & + & 4z & = & -4\,. \end{array}$$

9. An experiment consists in rolling two six-sided dice and adding the scores. What are the possible outcomes and their probabilities ?

Let A be the event that the outcome of the above experiment is an odd number. Let B be the event that the outcome is divisible by 3, and let C be the event that the outcome is not greater than 4.

Find:

(i) $P(A)$

(ii) $P(B)$

(iii) $P(C)$

(iv) $P(A \cup B)$

(v) $P(B|A)$

(vi) $P(C|A)$.

Are A and B independent events ? Are A and C independent events ? Are B and C independent events ?