UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION

MA1502 Algebra

Monday 31 May 2004 (12noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer all SIX questions in section A and TWO questions from section B.

Each question in section A is worth 10 marks and each question in section B is worth 20 marks.

SECTION A

1. **(a)** Find the real and imaginary parts and the complex conjugate of the complex number

$$\frac{2+3i}{1-4i}\,.$$

(b) Find the modulus and principal argument of each of the complex numbers

$$-1+i\,, \qquad 1-i\sqrt{3}\,, \qquad 5\cos\frac{\pi}{5}+5i\sin\frac{\pi}{5}.$$

2. Find the real and imaginary parts of $(1+i\sqrt{3})^{100}$.

3. Let A, B and C be points in 3–dimensional space with position vectors $\mathbf{a} = (1,2,3)$, $\mathbf{b} = (1,-1,0)$, $\mathbf{c} = (0,0,1)$, respectively. Calculate the following:

(i) $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})$;

(ii) the cosine of the angle at C of the triangle ABC;

(iii) $\mathbf{a}\times\mathbf{b}$;

(iv) the area of the triangle OAB;

(v) the value of λ such that $\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$ is perpendicular to $\mathbf{b}-\mathbf{a}$.

4. **(a)** Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$. Find all matrices $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 2 & 5 \\ 1 & 0 & -1 \end{bmatrix}$. Find $\det(A)$ and $\det(A^6)$.

5. **(a)** Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$. Write down the transpose A^T of A . Calculate AA^T.

(b) Let $C = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$. Find C^{-1} and C^2.

Use mathematical induction to show that $C^n = \begin{bmatrix} 1 & 1-2^{-n} \\ 0 & 2^{-n} \end{bmatrix}$ for all $n \geqslant 1$.

6. **(a)** Evaluate $\binom{14}{3}$ and $\binom{7}{4}$.

(b) A standard pack of cards contains $52 = 13 \times 4$ cards [13 ranks in four suits: the ranks are in increasing order 2,3,4,...,10, Jack, Queen, King, Ace and the suits are clubs, spades, hearts and diamonds.]

(i) Four cards are dealt from a well–shuffled pack. What is the probability that exactly two of them are kings ?

(ii) Two cards are dealt from a well–shuffled pack. What is the probability that they have the same rank ?

(iii) In a game for two players, one of the players (Player A) draws a single card from the well–shuffled pack without replacing it, then the other player (Player B) draws a single card from the remaining 51. It is agreed that player A wins if the rank of his/her card is $\geqslant$ the rank of the card drawn by player B. What is the probability that player A wins ?

SECTION B

7. **(a)** Show that $1 + 2i$ is a solution of the equation

$$x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$$

and find all the other solutions [real or complex] of this equation.

(b) Find all solutions of the equation $z^6 = -1$. For each solution, write down the modulus and principal argument.

8. **(a)** Let M be the plane with equation

$$2x + 4y - 3z = 4\,.$$

Write down a normal vector for this plane [that is, a vector which is perpendicular to the plane].

Let A and B be the points with position vectors $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (3, 0, 1)$, respectively. Find the vector equation of the straight line L through A and B. Show that L is parallel to the plane M and also to the xy–plane.

Find the distance from L to M and from L to the xy–plane.

(b) By using row reduction, find all solutions of the simultaneous equations

$$\begin{array}{ccccccc} x & - & 2y & + & z & = & 0 \\ x & & & + & 2z & = & -1 \\ x & + & 6y & + & 5z & = & -4\,. \end{array}$$

9. A sample of 976 households was surveyed regarding their laundry habits. Two questions asked were about the age of their washing machines, and the average intensity of use in cycles per week. [Households without washing machines were excluded from the survey.] The results were as follows.

	age < 2 yrs	age 2 to 5 yrs	age > 5 yrs	Total
< 2 cycles/week	34	87	91	212
2 to 4 cycles/week	124	206	143	473
> 4 cycles/week	133	122	36	291
Total	291	415	270	976

(a) One household is chosen at random from the 976. Let A be the event that the washing machine in that household is at least 2 years old. Let B be the event that the average number of washing cycles per week in that household is greater than 4.

Find:

(i) $P(A)$

(ii) $P(B)$

(iii) $P(A \cap B)$

(iv) $P(A \cup B)$

(v) $P(B|A)$

(vi) $P(A|B)$.

Determine whether A and B are independent events, and whether they are mutually exclusive or not.

(b) Now suppose that 10 households are chosen randomly from the 976, without replacement. What is the probability that exactly 5 of them have washing machines which are at least 2 years old ?