UNIVERSITY OF ABERDEEN **SESSION 2006-07**

DEGREE EXAMINATION

MA1502 Algebra

Monday 28 May 2007 (9 am to 11 am)

Answer all SIX questions in section A and TWO questions from section B.

Each question in section A is worth 10 marks. Each question in section B is worth 20 marks.

Marks may be deducted for answers which do not show clearly how the answer is reached.

The use of calculators in the exam is not allowed.

SECTION A

1. **(a)** Determine the real and imaginary parts of the complex number

$$\frac{3-7i}{2+6i} .$$

(b) Sketch the following complex numbers on an Argand diagram. (That means: depict them as "points in the plane".) For each of the four complex numbers, determine the modulus and the principal argument.

$$-2i\ , \quad -1+i\ , \quad \sqrt{3}+i\ , \quad \frac{1}{2}(\cos(\pi/6)+i\sin(\pi/6))\ .$$

2. **(a)** Find the real and imaginary parts of $(\sqrt{3}+i)^{-10}$.

(b) Determine the solutions of $z^2 = 16+30i$.

3. Three points in 3-space are given by $A=(1,2,3)$, $B=(4,0,-1)$ and $C=(-2,1,-3)$. Calculate the following:

(i) $\underline{AB}\cdot\underline{AC}$;

(ii) the cosine of the angle at A of the triangle ABC;

(iii) the coordinates of the point P on the line segment AB for which $|AP| = 2|PB|$;

(iv) the value of $\lambda\in\mathbb{R}$ such that $\underline{AB}+\lambda\underline{BC}$ is perpendicular to $\underline{AC}$.

4. **(a)** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & 3 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ -4 & 6 \end{bmatrix}$. Calculate AB.

(b) Let

$$A = \begin{bmatrix} 1/2 & x \\ y & -1/2 \end{bmatrix} .$$

Determine A^T , the transpose of A. For which choices of x and y is A^TA equal to the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ?$$

5. Let $f(n) = \sum_{j=1}^{n} j \cdot 2^j$ and $g(n) = (n-1)2^{n+1} + 2$.

(a) Calculate $f(1)$, $f(2)$ and $f(3)$.

(b) Calculate $g(1)$, $g(2)$ and $g(3)$.

(c) Use mathematical induction to show that $f(n) = g(n)$ for every integer $n \geqslant 1$.

6. **(a)** Evaluate $\binom{8}{4}$ and $\binom{14}{11}$.

(b) Make a list of all surjective maps from $\{1, 2, 3\}$ to $\{a, b\}$.

(c) Determine the number of surjective maps from the set $\{a, b, c, d, e\}$ to the set $\{1, 2, 3, 4\}$.

SECTION B

7. **(a)** You are told that $2 + i$ is a solution of the equation

$$x^4 - 3x^3 + 2x^2 + x + 5 = 0 \ .$$

Confirm this and find all the other solutions [real or complex] of this equation.

(b) Find all solutions of the equation $z^5 = -1 - i$. For each solution, write down the modulus and principal argument.

8. **(a)** Let M be the plane with equation

$$x_1 + 2x_2 - 3x_3 = 1 \,.$$

Determine a nonzero vector which is perpendicular to the plane.

Let $A = (1, 1, 5)$. Determine the coordinates of the point B on the plane M which has the shortest distance from A, and determine that distance.

(b) Construct an inverse for the matrix

$$\begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & 7 \\ 2 & -1 & -6 \end{bmatrix}$$

9. **(a)** Let X, Y, Z be sets. Let $g : X \to Y$ and $f : Y \to Z$ be maps. Suppose that g is surjective and f is not injective. Show that $f \circ g : X \to Z$ is not injective.

(b) A standard pack of cards contains $52 = 13 \times 4$ cards [13 ranks in four suits: the ranks are in increasing order 2,3,4,...,10, Jack, Queen, King, Ace and the suits are clubs, spades, hearts and diamonds.]

(i) Two cards are dealt from a well–shuffled pack. What is the probability that they are of different ranks ?

(ii) Four cards are dealt from a well–shuffled pack. Find an expression for the probability that at most two of them are Jacks.

(c) In how many ways can integers a_1, a_2, a_3, a_4 be selected, all $\geqslant 0$, such that

$$a_1 + a_2 + a_3 + a_4 = 11 \ ?$$