UNIVERSITY OF ABERDEEN **SESSION 2005–06**

DEGREE EXAMINATION

MA1002 Calculus

Tuesday 17 January 2006 (9 am to 11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Attempt ALL FIVE of the questions in SECTION A and TWO of the questions in SECTION B. Each question in section A is worth 12 marks and each question in section B is worth 20 marks.

SECTION A

1. Differentiate the following functions

$$f(x) = 3x^4 - 2x^{-1/3} + 6, \qquad g(x) = \frac{1-x}{1+x+x^2},$$

$$h(x) = xe^{-x^2+2x}, \qquad k(x) = \sin(\sqrt{1+x^2}).$$

2. **(a)** Find the indefinite integrals

$$\int (x^3 - 3x^{-3} - x^{-1})\,dx, \qquad \int \left(\sin(2x) - \frac{1}{1+x^2}\right)\,dx.$$

(b) Evaluate the definite integrals

$$\int_0^1 \frac{x}{x^2+3}\,dx, \qquad \int_0^\pi \sin(t)e^{-t}\,dt.$$

3. **(a)** Where does the graph of $y = e^x$ cross the line $y = 2$? Sketch the finite region bounded by the lines $x = 0$, $y = 2$ and the graph $y = e^x$. Show that the region has area $\ln(4) - 1 \approx 0.39$.

(b) What is the volume of the solid of revolution obtained by taking the part of the graph of $y = e^{-x}$ between $x = 0$ and $x = 1$ and rotating it about the x-axis?

4. **(a)** Find the maximum and minimum values of $f(x) = x^2(x^2-3)$ on the interval $-2 \leqslant x \leqslant 2$.

(b) Where does $x + \dfrac{1}{x^2}$ achieve its minimum value for $x > 0$?

5. By sketching the graphs $y = x$ and $y = e^{-x}$, explain why the equation $x = e^{-x}$ only has one solution and use the Newton-Raphson method to find an approximation to this solution, stating your result accurate to 4 decimal places. Use $x = 0.5$ as your first guess.

SECTION B

6. **(a)** Suppose that $Ae^x + Be^{-x} = 0$ for all values of x, where A and B are constants. By differentiating the left-hand side, or otherwise, show that $A = B = 0$.

(b) Let $y(x) = e^{-3x} \sin px$, where p is a constant. Show that

$$y'' = e^{-3x} \left\{ (9 - p^2) \sin px - 6p \cos px \right\}.$$

For what values of p does $y(x)$ satisfy the differential equation $y'' + 6y' + 10y = 0$?

7. **(a)** Show that

$$\int_0^1 \frac{3x - 1}{(x + 1)(x + 2)(x + 3)}\, dx = 12 \times \ln 3 - 19 \times \ln 2.$$

(b) Find the solution of the equation $x^2 \dfrac{dy}{dx} = 1 + y^2$ which satisfies y=0 when x=1.

8. **(a)** Find the equation of the tangent to the curve given by $x^4 - 2x^2y + y^4 = 1$ at the point $(0, 1)$. Where else does this tangent line meet the curve?

(b) Prove that the derivative of the function $\arccos(x)$ exists for $-1 < x < 1$ and is given by

$$\frac{-1}{\sqrt{1 - x^2}}.$$

You may assume a knowledge of the derivative of the cosine function.

9. A window consists of a semi-circle above a rectangle as shown. The distance round the edge of the window is 10 metres. Obtain a formula for the area of the window in terms of the width and height of the rectangle. What is the ratio of the width to the height of the rectangle when the area of the window is as large as possible?