UNIVERSITY OF ABERDEEN **SESSION 2006–2007**

DEGREE EXAMINATION

ES 3006 Engineering Analysis and Methods 1B

Tuesday 16 January 2007 (2 pm to 5 pm)

NOTES: (i) You **must not** have in your possession any material other than that expressly permitted in the rules appropriate to this examination. Where this is permitted, such material **must not** be amended, annotated or modified in any way.

(ii) You **must not** have in your possession any material that could be determined as giving you an advantage in the examination.

(iii) You **must not** attempt to communicate with any candidate during the examination, either orally or by passing written material, or by showing material to another candidate, nor must you attempt to view another candidate's work.

Failure to comply with the above will be regarded as cheating and may lead to disciplinary action as indicated in the Academic Quality Handbook: `www.abdn.ac.uk/registry/quality/appendix7x1.pdf` **(Sections 4.14 and 5).**

(iv) Candidates are permitted to use approved calculators.

(v) Candidates are permitted to use the Engineering Mathematics Handbook, which will be provided for them.

(vi) The specification of some MATLAB commands is appended for use in questions 6 and 7.

Candidates should attempt FIVE questions. All questions carry 20 marks.

1. **(a)** Let Γ be the curve given by $\mathbf{r} = (u^4 - 1,\, u^2 - 1,\, e^{u+1})$. Find a vector equation of the line tangent to Γ at the point with parameter $u = -1$.

[4 marks]

(b) Let Σ be the surface given by $\mathbf{r} = (u + 3v,\, 1 + uv,\, v^2)$. Find a unit normal to Σ at the point (4, 2, 1). Find a coordinate equation of the plane tangent to the surface Σ at the same point.

[8 marks]

(c) Let P be a particle moving in space and suppose that the position vector $\mathbf{r}$ of P relative to axes with origin O is given by

$$\mathbf{r} = (3 - t^2,\, 3 + 2t^2,\, 1 - 3t^2),$$

where t is the time from a fixed instant. Calculate the velocity, speed and acceleration of P expressing your answers as functions of t. What distance does P travel in the time interval from $t = 0$ to $t = 1$?

[8 marks]

2. **(a)** The heat flux vector field in a region of space has the direction of the maximum rate of *decrease* in temperature at each point. Suppose that the temperature in a region is given by $u = 3x^2 + 2y^2 + z^2$. Find the unit vector in the direction of the heat flux vector field at the point (0, 0, 1).

[4 marks]

(b) Find a coordinate equation for the tangent plane to the surface $x^2 + 4y^2 + z^2 = 6$ at the point (1, 1, 1).

[4 marks]

(c) Let ϕ be the scalar field given by $\phi = \dfrac{1}{|\mathbf{r}|}$ where $\mathbf{r} = (x,\ y,\ z)$. Show that

$$\operatorname{grad}\phi = -\frac{1}{|\mathbf{r}|^3}\,\mathbf{r}.$$

[4 marks]

(d) Let $\mathbf{v}$ be a vector field and let ϕ be a scalar field, both defined in space. Show that

$$\operatorname{div}(\phi\mathbf{v}) = \operatorname{grad}\phi \cdot \mathbf{v} + \phi \operatorname{div}\mathbf{v}.$$

Also show that $\operatorname{curl}\operatorname{grad}\phi = 0$. [8 marks]

3. **(a)** Let $\mathbf{v} = (1+y+z+yz,\ 1+x+z+xz,\ 1+y+x+xy)$. It is given that $\mathbf{v}$ is a conservative vector field (you are *not* required to show this). Find a potential for $\mathbf{v}$.

Evaluate $\displaystyle\int_\Gamma \mathbf{v} \cdot d\mathbf{r}$ where Γ is the line segment from (1, 1, 1) to (2, 3, 0).

[8 marks]

(b) Let $\mathbf{w}$ be the vector field given by $\mathbf{w} = (z - x^2y, x - y^2z,\ y - z^2x)$. Show that $\mathbf{w}$ is not a conservative vector field.

[4 marks]

(c) Evaluate $\displaystyle\iint_T xy^2\,dx\,dy$ where T is the triangular region in the xy-plane enclosed by the lines $y = x$, $y = 1$ and $x = 0$.

[8 marks]

/OVER

4. **(a)** Let $\mathbf{v}$ be the vector field in the xy-plane given by $\mathbf{v} = (3x^2y + 6y + 1,\ x^3 + 4x + y^2)$ and let C be the circle centered at the point (1, 1), with radius 2 and with the positive orientation. Draw a sketch of C indicating the orientation of C.

Evaluate $\displaystyle\int_C \mathbf{v} \cdot d\mathbf{r}$.

[10 marks]

(b) Are the two graphs, represented by the diagrams in Figure 1, isomorphic? Justify your answer.

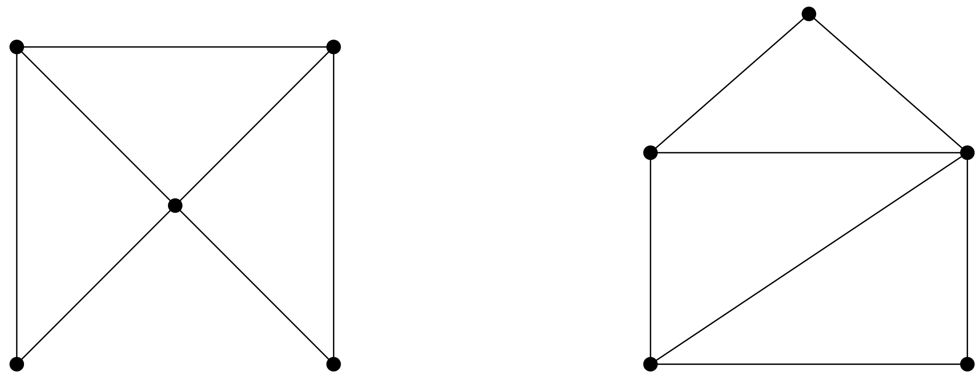

Figure 1 (Question 4 (b))

[10 marks]

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5. **(a)** The scheduling network for a construction project is shown in Figure 2. The weights are the completion times in days for the tasks of the project. Determine the minimum completion time, the float times of each vertex and a critical path.

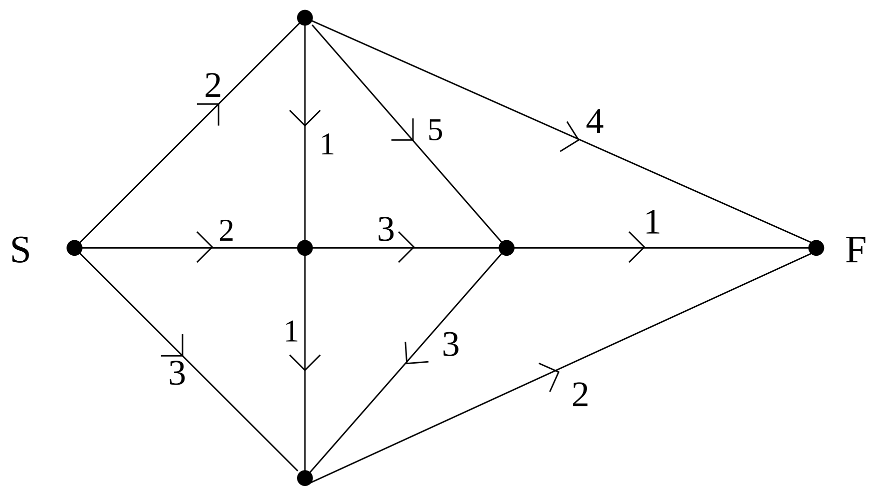

Figure 2 (Question 5 (a))

[10 marks]

(b) Find the value of a maximal flow for the network represented by Figure 3. Indicate in a diagram your maximal flow and a minimal cut.

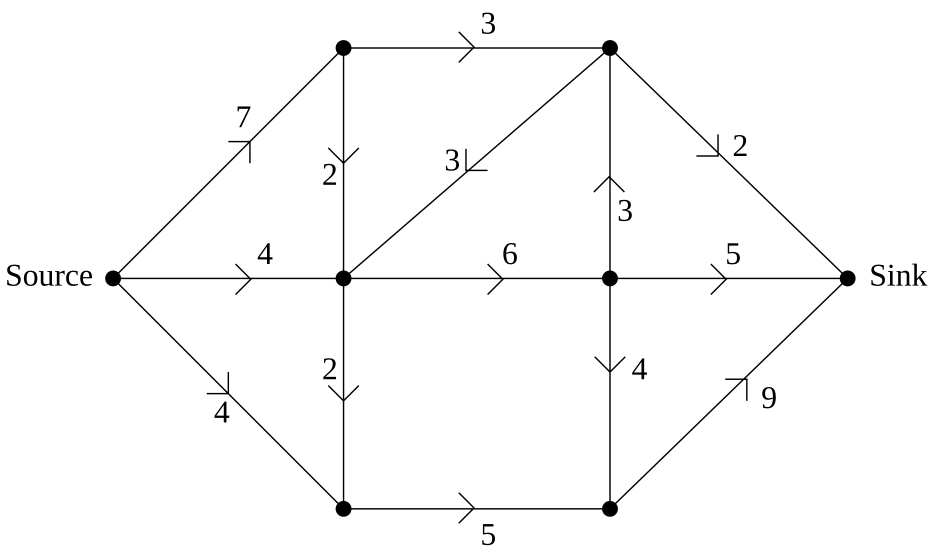

Figure 3 (Question 5 (b))

[10 marks]

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6. **(a)** Write MATLAB statements using `diff` and `subs` which will obtain an expression for the derivative dy/dx of

$$y = x^4 - 5x^3 + 1$$

and evaluate the derivative when $x = 5$.

[5 marks]

(b) Write MATLAB statements involving `dsolve` which will give an exact (analytic) solution of the initial value problem:

$$\frac{dx}{dt} - 2x = 5\sin(t), \qquad \text{with } x = 3 \text{ when } t = 2.$$

[5 marks]

(c) Write a MATLAB `function` M-file `f.m` to define the function

$$f(t, x) = t \cdot \sin(x)\,.$$

Using the routine `ode45`, give a MATLAB statement that produces a numerical solution of the initial value problem

$$\frac{dx}{dt} = t \cdot \sin(x) \qquad \text{with } x = 1 \text{ when } t = 0,$$

in the range $0 \le t \le 3$.

[10 marks]

7. Consider the heat equation

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2},$$

for $t \geqslant 0$ and $-2 \leqslant x \leqslant 2$, subject to the homogeneous boundary conditions

$$u(t, -2) = 0, \quad u(t, 2) = 0 \quad \text{for } t \geqslant 0.$$

(i) Show that for any integer n, $u = e^{-n^2\pi^2 t/2} \sin\left(\frac{n\pi x}{2}\right)$ satisfies the heat equation and the homogeneous boundary conditions.

Write down the solution of the heat equation subject to the homogeneous boundary conditions and, in addition, the initial condition

$$u(0,\, x) = 2\sin(\pi x) + \sin(2\pi x).$$

[8 marks]

(ii) Provide MATLAB code using the routine `pdepe` to obtain a numerical solution of the heat equation above for $0 \le t \le 1$, $-2 \le x \le 2$, subject to the given homogeneous boundary conditions and, in addition, the initial condition

$$u(0, x) = 4 - x^2.$$

You should specify all the `function` M-files required.

[12 marks]