UNIVERSITY OF ABERDEEN **SESSION 2005-2006**

Degree Examination

EG2510 Engineering Mathematics 3

Tuesday 30 May 2006 (3 pm—5 pm)

NOTES *(i) Candidates are permitted to use approved calculators.*

(ii) Candidates are permitted to use the Engineering Mathematics Handbook which will be provided for them.

Candidates should attempt ALL FOUR questions in SECTION A and TWO of the questions in SECTION B.

SECTION A

1. **(a)** Find the eigenvalues of the matrix $\begin{bmatrix} 7 & -5 \\ 4 & -2 \end{bmatrix}$ and calculate, for each eigenvalue, a corresponding eigenvector.

[5 marks]

(b) You are given, and are NOT asked to check, that the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

has eigenvalues 1 and 5 with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ respectively.

From this information, write down an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Variables x_1 and x_2 are functions of a variable t. Write down the general solution of the first order system

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_1 + 3x_2 \\ \frac{dx_2}{dt} &= x_1 + 2x_2 \end{aligned}$$

Hence find the solution of the system that satisfies the initial condition: $x_1 = 10$ and $x_2 = 2$ when $t = 0$.

[5 marks]

2. **(a)** A box contains 10 resistors, of which 3 are faulty. A sample of 2 is chosen at random.

(i) What is the probability that none of the resistors in the sample is faulty?

(ii) What is the probability that at least one of the resistors in the sample is faulty?

[4 marks]

/continued

(b) A coin is biased in such a way that the probability of obtaining heads in a single toss is 0.6. The coin is thrown 5 times.

(i) Find the probability of getting exactly 3 heads.

(ii) Find the probability of getting at least 3 heads.

What is the probability of getting heads for the first time on the 4th throw?

[6 marks]

3. Values of a function $f(x)$ are found by experiment to be:

x :	1.5	1.6	1.7	1.8	1.9	2.0
$f(x)$:	203	264	361	490	647	828

Estimate the value of $f(1.73)$ by using (i) linear interpolation and (ii) Bessel's second order interpolation formula.

Use inverse linear interpolation to estimate the value of x (in the given range) for which $f(x) = 447$.

[10 marks]

4. **(a)** Find the least squares best fit straight line to the data points:

x :	0.0	0.1	0.2	0.3	0.4
y :	977	782	576	384	176

[6 marks]

(b) A real-valued function $f(x)$ has the property that $|f''(x)| \leq 2$ for all values of x in the range considered. Given that $f(5) = 3$ and $f'(5) = 1$, estimate, as accurately as you can, the value of $f(5.02)$. What is the maximum possible error in your estimate?

[4 marks]

SECTION B

5. **(a)** Consider the real symmetric matrix $A = \begin{bmatrix} 18 & 6 \\ 6 & 34 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

Deduce that the matrix A is positive-definite, that is, show that $\mathbf{x}^T A\mathbf{x} > 0$ for every non-zero real vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Variables x_1 and x_2 are functions of a variable t (time). Determine the normal modes of the second order system

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -18x_1 - 6x_2, \\ \frac{d^2x_2}{dt^2} &= -6x_1 - 34x_2, \end{aligned}$$

/continued

giving the (angular) frequencies of the normal modes. Hence, write down the general solution of the system.

[14 marks]

(b) The matrix

$$\begin{bmatrix} 7 & -15 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 5 \end{bmatrix}$$

has 3 distinct real eigenvalues. Determine the eigenvalues and find, for each of the eigenvalues, a corresponding eigenvector.

[6 marks]

6. **(a)** A manufacturer buys a certain component from two suppliers M and N, obtaining 60% from M and 40% from N. Records indicate that 3% of the components supplied by M are faulty and 4% of those supplied by N are faulty.

A component is chosen at random from stock. Let A denote the event that the component was supplied by M, and let B denote the event that the component is faulty.

Write down the probability $\mathbb{P}(A)$ and the conditional probability $\mathbb{P}(B \mid A)$.

Calculate the probabilities

$$\text{(i)} \quad \mathbb{P}(A \cap B); \quad \text{(ii)} \quad \mathbb{P}(B); \quad \text{(iii)} \ \mathbb{P}(A \mid B).$$

[6 marks]

(b) A continuous random variable X has probability density function

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{1}{10} & \text{if } 0 \le x \le 10, \\ 0 & \text{if } x > 10. \end{cases}$$

Sketch the graph of $f_X(x)$ against x.

What is the probability $\mathbb{P}(5 < X \le 7)$?

What is the conditional probability $\mathbb{P}(X \le 7 \mid X > 5)$?

Write down the expectation $\mathbb{E}(X)$, and calculate the variance var(X) and standard deviation $\mathrm{SD}(X)$ of X.

[7 marks]

(c) At a certain location, the waiting time T (seconds) for the next click on a Geiger counter due to background radiation has the exponential probability density function

$$f_T(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{10} e^{-t/10} & \text{if } t \ge 0. \end{cases}$$

(i) Write down the value of the expectation, $\mathbb{E}(T)$, of T.

(ii) Determine the probability $\mathbb{P}(T > 15)$ that the time before the next click is more than 15 seconds.

(iii) What is the probability that the next click will occur in less than 5 seconds?

[7 marks]

7. **(a)** Consider the second order initial value problem:

$$\ddot{x} + \dot{x} + x^3 = 1 - t^2, \qquad x = 1,\ \dot{x} = 2 \text{ when } t = 0.$$

Write down an equivalent first order system and use two steps of Euler's method (with step length 0.01) to estimate the values of x and $\dot{x}$ when $t = 0.02$.

[10 marks]

(b) Find the value of the constant $\lambda > 0$ such that

$$\int_{-h}^{h} f(x)\,dx = h(f(-\lambda h) + f(\lambda h))$$

for any function $f(x)$ that is a polynomial of degree less than or equal to 3 (and any step length $h > 0$).

For this value of λ, determine the error

$$\left| \int_{-h}^{h} f(x)\,dx - h(f(-\lambda h) + f(\lambda h)) \right|$$

in using this numerical integration formula when $f(x)$ is a polynomial of degree 4 in terms of $M = |f^{(4)}(0)|$ and h.

[10 marks]