UNIVERSITY OF ABERDEEN **SESSION 2004-2005**

DEGREE EXAMINATION

EG2510 Engineering Mathematics 3

Monday 23 May 2005 (9 am—11 am)

NOTES **(i) Candidates are permitted to use approved calculators.**

(ii) Candidates are permitted to use the Engineering Mathematics Handbook which will be provided for them.

Candidates should attempt ALL FOUR questions in SECTION A and TWO of the questions in SECTION B.

SECTION A

1. **(a)** Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Find an invertible matrix $\mathbf{X}$ such that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is a diagonal matrix. Write down the diagonal matrix.

[10 marks]

2. **(a)** A bag contains 3 red balls and 5 blue balls. Five balls are chosen at random from the bag and placed in a box.

(i) Find the probability that all of the balls in the box are blue.

(ii) Find the probability that exactly one of the balls in the box is red.

(iii) Find the probability that at least two of the balls in the box are red.

[6 marks]

(b) A fair coin is thrown 8 times. The coin lands 'heads' or 'tails'.

(i) Find the probability that 'heads' occurs in exactly three of the throws.

(ii) Find the probability that the first throw is a 'head' and there are exactly two more 'heads' thrown.

[4 marks]

3. Values of a function $f(x)$ are found by experiment to be:

x :	0.0	0.1	0.2	0.3	0.4	0.5
$f(x)$:	314	397	503	630	776	939

Estimate the value of $f(0.26)$ by using (i) linear interpolation and (ii) Bessel's second order interpolation formula.

Use Bessel's second order formula to estimate the value of x (in the given range) for which $f(x) = 539$.

[10 marks]

4. **(a)** An experiment gives the following data points:

x :	0	1	2	3	4
z :	1.164	1.415	1.735	2.115	2.588

It is known that variables z and x are related by an equation of the form $z = A\mathrm{e}^{mx}$, where A and m are constants and $A > 0$. Taking logarithms, the formula becomes $\ln(z) = mx + \ln(A)$. Use a least squares best fit straight line to the data points $(x, \ln(z))$, which have been tabulated below, to estimate the values of m and A.

x :	0	1	2	3	4
$\ln(z)$:	0.152	0.347	0.551	0.749	0.951

[7 marks]

(b) A real-valued function $f(x)$ has the property that $|f'(x)| \leq 3$ for all values of x in the range considered. Given that $f(2) = 5.4$, what is the maximum possible error in estimating $f(1.999)$ as 5.400 ?

[3 marks]

SECTION B

5. **(a)** Two wagons of masses m and $2m$ are connected by a spring coupling with spring constant $4k$. One of the wagons is attached by a spring coupling with spring constant $5k$ to a fixed point as in Figure 1. The track is horizontal and frictional forces are negligible.

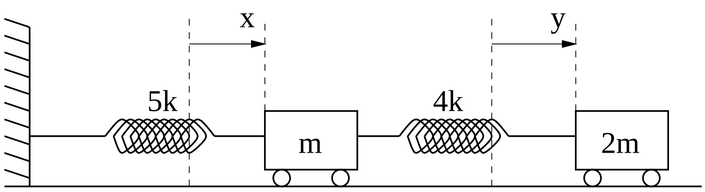

Figure 1 (Question 5)

Let x and y be the displacements of the wagons from their equilibrium positions as indicated in Figure 1 (x and y depend on time t). Show that

$$\begin{aligned} \ddot{x} &= ax + by, \\ \ddot{y} &= cx + dy, \end{aligned}$$

where a, b, c and d are constants which should be determined. [*You are NOT required to solve this system.*]

[8 marks]

(b) Let $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 5 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. By evaluating $\mathbf{Ax}_1$, $\mathbf{Ax}_2$ and $\mathbf{Ax}_3$, or otherwise, show that $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are eigenvectors of $\mathbf{A}$.

Find an orthogonal matrix $\mathbf{Y}$ such that $\mathbf{Y}^T\mathbf{AY}$ is a diagonal matrix.

By considering $\mathbf{x}^T\mathbf{Ax}$ where $\mathbf{x} = [x\ y\ z]^T$, or otherwise, show that

$$3x^2 + 4y^2 + 5z^2 + 4xy + 4yz \geqslant 0$$

for all values of x, y and z.

[12 marks]

6. **(a)** A factory assembles television sets, working 5 days a week and producing the same number of sets each day. Checks over a long period have shown that 5% of the sets produced on a Friday are faulty. On the other working days only 3% are faulty. Suppose that a set is chosen at random from a week's output. Let A be the event that the set was assembled on Friday and let B be the event that the set is faulty.

Write down the probability $P(A)$ and the conditional probability $P(B|A)$, and calculate $P(A \cap B)$.

Using a tree diagram, or otherwise, calculate the probabilities (i) $P(B)$, (ii) $P(A|B)$.

[6 marks]

(b) The lifetime, in hours, of a certain type of electronic component is a continuous random variable T with exponential probability density function

$$f_T(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{2000} e^{-t/2000} & \text{if } t \geq 0. \end{cases}$$

(i) Write down the value of $E(T)$, the expectation of T.

(ii) Determine the probability that a given component will function for more than 3000 hours.

(iii) What percentage of the components will fail in less than 3000 hours?

[7 marks]

(c) A continuous random variable X has probability density function

$$f_X(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ x & \text{if } 0 < x \leqslant 1, \\ 2 - x & \text{if } 1 < x \leqslant 2, \\ 0 & \text{if } x > 2. \end{cases}$$

Sketch the graph $y = f_X(x)$.

Give the expectation $E(X)$ and calculate the variance $\text{var}(X)$ of X.

[7 marks]

7. **(a)** Consider the initial value problem:

$$\ddot{x} = -x - (x^2 - 1)\dot{x}, \qquad x = -2,\ \dot{x} = 3 \text{ when } t = 0.$$

Write down an equivalent first order system and use two steps of Euler's method (with step length 0.01) to estimate the values of x and $\dot{x}$ when $t = 0.02$.

[10 marks]

(b) Find values of constants A, B and C such that

$$\int_{-2h}^{2h} f(x)\,dx = h(Af(-2h) + Bf(-h) + Cf(0) + Bf(h) + Af(2h))$$

for any function $f(x)$ that is a polynomial of degree less than or equal to 5 (and any step length $h > 0$).

[10 marks]