UNIVERSITY OF ABERDEEN **SESSION 2004–2005**

Degree Examination

EG2010 Engineering Mathematics 2

Tuesday 18 January 2005 (3pm to 5pm)

NOTES:

(i) Candidates are permitted to use approved calculators.

(ii) Candidates are permitted to use the Engineering Mathematics Handbook, which will be available to them.

Candidates should attempt ALL questions from Section A and any TWO questions from section B.

SECTION A

1. **(a)** Find the general solution of the ordinary differential equation

$$\frac{dx}{dt} + 3x = 9t - 6,$$

using a particular integral of the form $x = pt + q$. [5 marks]

(b) Solve the initial value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 0, \quad \text{with } x = 5 \text{ and } \frac{dx}{dt} = -3 \text{ when } t = 0.$$

[5 marks]

2. **(a)** A variable u is given as a function of two variables x, y by

$$u = x\mathrm{e}^{xy} + x^2y.$$

Find the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ and verify that

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u.$$

[4 marks]

(b) The function u of the three variables x, y and z is given by $u = \mathrm{e}^{cx}\sin(4y)\cos(3z)$, where c is a constant. Find the two values of c for which u satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

[6 marks]

3. **(a)** Find the two critical points of the function

$$f(x, y) = x^2 + y^3 - y^2 - y$$

and classify them as local maxima, local minima or saddle points. [5 marks]

(b) Find the one critical point of the function

$$f(x, y) = e^{xy+y},$$

and classify that critical point as a local maximum, a local minimum or a saddle point.

[5 marks]

4. **(a)** Find the Laplace transform of each of the following functions

(i) $f(t) = 4t^3 + 3t^2 - 5$,

(ii) $f(t) = e^{-3t} + 2\sin(4t)$,

(iii) $f(t) = e^{2t}\cos(t)$.

[5 marks]

(b) Find the inverse Laplace transform of each of the following functions

(i) $F(s) = \dfrac{6}{(s+2)(s-1)}$,

(ii) $F(s) = \dfrac{s+6}{s^2+4}$.

[5 marks]

SECTION B

5. **(a)** Find the general solution of the homogeneous linear differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 0.$$

Classify the system described by this differential equation as under damped, critically damped or over damped. Give the value of the natural (angular) frequency and the damping parameter. [You are reminded of the theoretical expression $2k\omega_n \frac{dx}{dt}$.] [7 marks]

(b) Using your result from (a), find the general solution of the non-homogeneous equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 30\cos(2t).$$

Describe the behaviour of the solution as $t \to \infty$. Write down the steady state response and determine its amplitude. [13 marks]

6. **(a)** The period T of small oscillations of a weight of mass m suspended on a (weightless) spring of spring constant k is given by $T = 2\pi\sqrt{\frac{m}{k}}$ so that $k = 4\pi^2 \frac{m}{T^2}$. This formula is used to measure the spring constant k. The mass m can only be measured to an accuracy of $\pm 1.5\%$ and the period T to an accuracy of $\pm 2\%$. Find the approximate maximum percentage error in the resulting measurement of k. [9 marks]

(b) A variable u is a function of the Cartesian coordinates (x, y), where $x > 0$. The polar coordinates (r, θ) are related to the Cartesian coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Using the chain rule, express the partial derivative $\frac{\partial u}{\partial \theta}$ first of all in terms of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, r and θ and then in terms of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, x and y. Express the partial derivative $\frac{\partial u}{\partial r}$ similarly. By using one of the equations that you have obtained, or otherwise, transform the partial differential equation

$$-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

into polar coordinates, and find its general solution. If, in addition, $u(x, 0) = 1 + x^2$ for all $x > 0$, determine the solution satisfying this initial condition. [11 marks]

7. **(a)** An odd function $f(x)$ is periodic with period 2π. It is given in the range $0 \le x \le \pi$ by:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < \pi \\ 0 & \text{if } x = \pi. \end{cases}$$

Sketch the graph of $f(x)$ in the range $-\pi \le x \le 3\pi$.

Determine the Fourier sine series of $f(x)$. [8 marks]

(b) Consider the initial value problem:

$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = -x - 3y + \mathrm{e}^{-2t},$$

with $x = 0$ and $y = 0$ when $t = 0$.

Show that the Laplace transform $X(s)$ of x is given by

$$X(s) = \frac{1}{(s+2)^3}.$$

By determining the inverse Laplace transform of $X(s)$, express x as a function of t. Hence, or otherwise, express y as a function of t.

[12 marks]