THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

Specimen Test One – Issued May 2006

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS

NAME:

TEST CENTRE:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH: **Special Arrangements: [ ]**

NOTE: This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

{ **Mathematics**, **Maths & Philosophy**, **Maths & Statistics** } candidates should attempt **Questions 1, 2, 3, 4, 5**.

Maths & Computer Science candidates should attempt **Questions 1, 2, 3, 5, 6.**

Computer Science candidates should attempt **Questions 1, 2, 5, 6, 7.**

Further credit cannot be gained by attempting extra questions.

Question 1 is a multiple choice question with ten parts, for which marks are given solely for the correct answers, though you may use the space between parts for rough work. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to Questions 2–7 should be written in the space provided, continuing onto the blank pages at the end of this booklet if necessary. Each of Questions 2–7 is worth 15 marks.

ONLY ANSWERS WRITTEN IN THIS BOOKLET WILL BE MARKED.
DO NOT INCLUDE EXTRA SHEETS OR ROUGH WORK.

THE USE OF CALCULATORS, FORMULA SHEETS
AND DICTIONARIES IS PROHIBITED.

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The area of the region bounded by the curves $y = x^2$ and $y = x + 2$ equals

(a) $\frac{7}{3}$ (b) $\frac{7}{2}$ (c) $\frac{9}{2}$ (d) $\frac{11}{2}$

B. The smallest value of the function

$$f\,(x) = 2x^3 - 9x^2 + 12x + 3$$

in the range $0 \leq x \leq 2$ is

(a) 1 (b) 3 (c) 5 (d) 7

C. What is the reflection of the point $(3,4)$ in the line $3x + 4y = 50$?

(a) $(9,12)$ (b) $(6,8)$ (c) $(12,16)$ (d) $(16,12)$

D. The equation $x^3 - 30x^2 + 108x - 104 = 0$

(a) no real roots;
(b) exactly one real root;
(c) three distinct real roots;
(d) a repeated root.

E. The fact that

$$6 \times 7 = 42,$$

is a counter-example to which of the following statements?

(a) the product of any two odd integers is odd;

(b) if the product of two integers is not a multiple of 4 then the integers are not consecutive;

(c) if the product of two integers is a multiple of 4 then the integers are not consecutive;

(d) any even integer can be written as the product of two even integers.

F. How many values of x satisfy the equation

$$2\cos^2 x + 5\sin x = 4$$

in the range $0 \leqslant x < 2\pi$?

(a) 2 (b) 4 (c) 6 (d) 8

G. The inequalities $x^2 + 3x + 2 > 0$ and $x^2 + x < 2$, are met by all x in the region:

(a) $x < -2$;
(b) $-1 < x < 1$;
(c) $x > -1$;
(d) $x > -2$.

H. Given that

$$\log_{10} 2 = 0.3010 \text{ to 4 d.p. and that } 10^{0.2} < 2$$

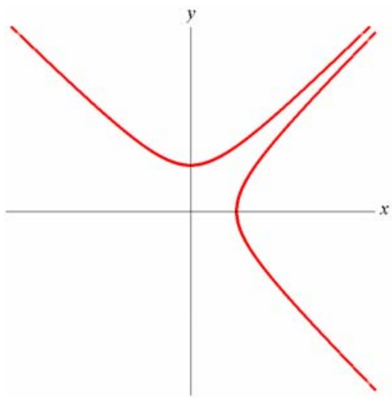
it is possible to deduce that

(a) 2^{100} begins in a 1 and is 30 digits long;
(b) 2^{100} begins in a 2 and is 30 digits long;
(c) 2^{100} begins in a 1 and is 31 digits long;
(d) 2^{100} begins in a 2 and is 31 digits long.

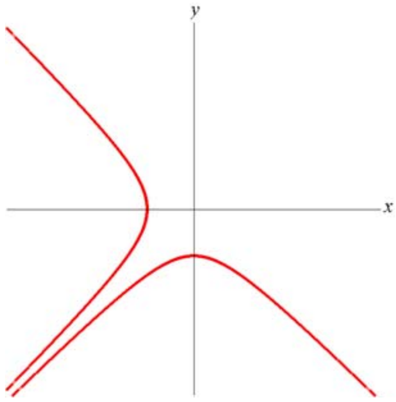
I. The power of x which has the greatest coefficient in the expansion of $\left(1 + \frac{1}{2}x\right)^{10}$ is

(a) x^2 (b) x^3 (c) x^5 (d) x^{10}

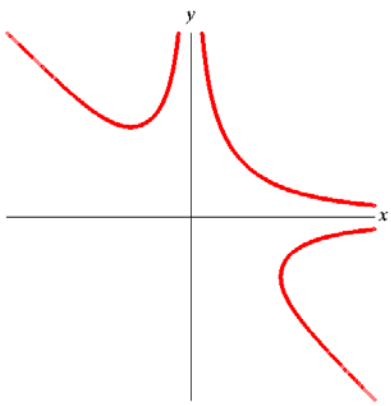
J. A sketch of the curve with equation $x^2y^2\,(x + y) = 1$ is drawn in which of the following diagrams?



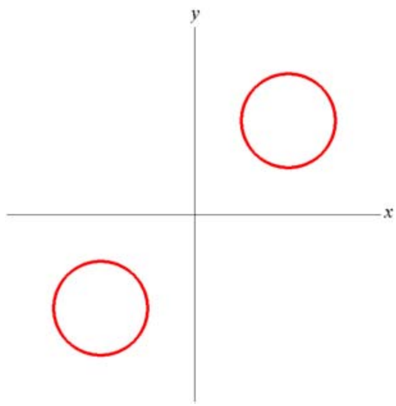
(a)



(b)



(c)



(d)

2. For **ALL APPLICANTS.**

(i) Show, with working, that

$$x^3 - (1 + \cos\theta + \sin\theta)\, x^2 + (\cos\theta \sin\theta + \cos\theta + \sin\theta)\, x - \sin\theta \cos\theta, \tag{1}$$

equals

$$(x - 1)\left(x^2 - (\cos\theta + \sin\theta)\, x + \cos\theta \sin\theta\right)$$

Deduce that the cubic in (1) has roots

$$1, \qquad \cos\theta, \qquad \sin\theta.$$

(ii) Give the roots when $\theta = \frac{\pi}{3}$.

(iii) Find all values of θ in the range $0 \leqslant \theta < 2\pi$ such that two of the three roots are equal.

(iv)What is the greatest possible difference between two of the roots, and for what values of θ in the range $0 \leqslant \theta < 2\pi$ does this greatest difference occur?

Show that for each such θ the cubic (1) is the same.

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

In this question we shall consider the function $f(x)$ defined by

$$f(x) = x^2 - 2px + 3$$

where p is a constant.

(i) Show that the function $f(x)$ has one stationary value in the range $0 < x < 1$ if $0 < p < 1$, and no stationary values in that range otherwise.

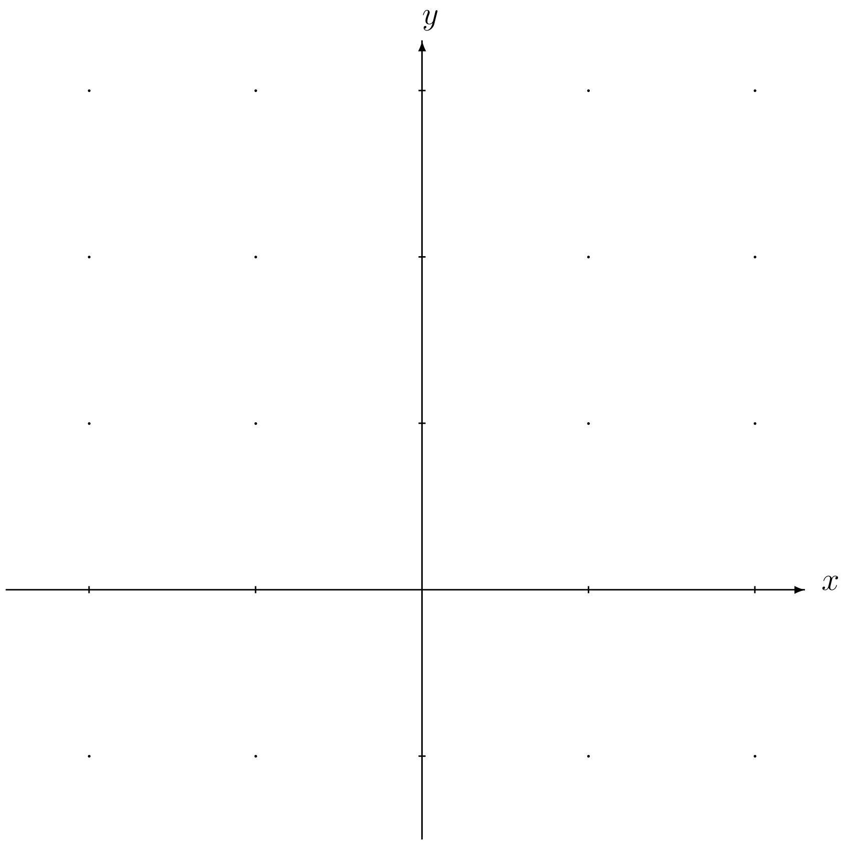
In the remainder of the question we shall be interested in the smallest value attained by $f(x)$ in the range $0 \leqslant x \leqslant 1$. Of course, this value, which we shall call m, will depend on p.

(ii) Show that if $p \geqslant 1$ then $m = 4 - 2p$.

(iii) What is the value of m if $p \leqslant 0$?

(iv) Obtain a formula for m in terms of p, valid for $0 < p < 1$.

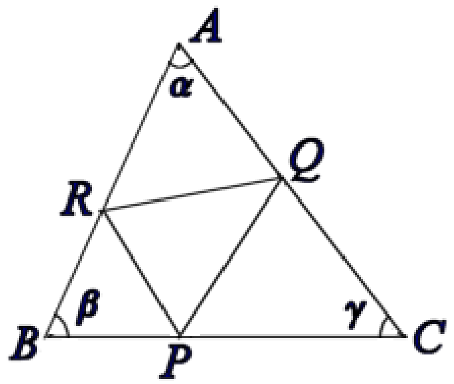
(v)Using the axes opposite, sketch the graph of m as a function of p in the range $-2 \leqslant p \leqslant 2$.

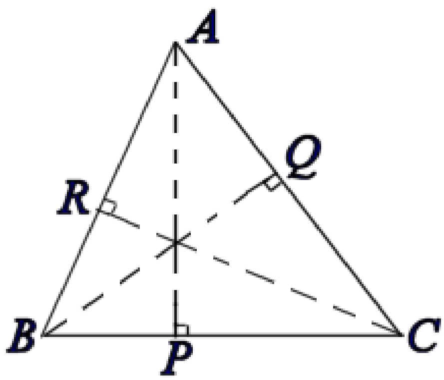

y
x

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Maths & Computer Science **and** *Computer Science* **applicants should turn to page 14.**





A triangle ABC has sides BC, CA and AB of sides a, b and c respectively, and angles at A, B and C are α, β and γ where $0 \leqslant \alpha, \beta, \gamma \leqslant \frac{1}{2}\pi$.

(i) Show that the area of ABC equals $\frac{1}{2}bc\sin\alpha$.

Deduce the sine rule

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

(ii) The points P, Q and R are respectively the feet of the perpeniculars from A to BC, B to CA, and C to AB as shown.

Prove that

$$\text{Area of } PQR = \left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\right) \times (\text{Area of } ABC).$$

(iii) For what triangles ABC, with angles α, β, γ as above, does the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

hold?

5. For **ALL APPLICANTS.**

Songs of the Martian classical period had just two notes (let us call them x and y) and were constructed according to rigorous rules:

I. the sequence consisting of no notes was deemed to be a song (perhaps the most pleasant);

II. a sequence starting with x, followed by two repetitions of an existing song and ending with y was also a song;

III. the sequence of notes obtained by interchanging xs and ys in a song was also a song.

All songs were constructed using those rules.

(i) Write down four songs of length six (that is, songs with exactly six notes).

(ii) Show that if there are k songs of length m then there are $2k$ songs of length $2m+2$. Deduce that for each natural number there are 2^n songs of length $2^{n+1}-2$.

Songs of the Martian later period were constructed using also the rule:

IV. if a song ended in y then the sequence of notes obtained by omitting that y was also a song.

(iii) What lengths do songs of the later period have? That is, for which natural numbers n is there a song with exactly n notes? Justify your answer.

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Alice, Bob and Charlie are well-known expert logicians.

(i) The King places a hat on each of their heads. Each of the logicians can see the others' hats, but not his or her own.

The King says "Each of your hats is either black or white, but you don't all have the same colour hat".

All four are honest, and all trust one another.

The King now asks Alice "Do you know what colour your hat is?".

Alice says "Yes, it's white".

What colour are the others' hats? [Hint: think about how Alice can deduce that her hat is white.]

(ii) The King now changes some of the hats, and again says "Each of your hats is either black or white, but you don't all have the same colour hat". He now asks Alice "Do you know what colour your hat is?".

Alice replies "No"

Can Bob work out what colour his hat is? Explain your answer. [Hint: what can Bob deduce from the fact that Alice can't tell what colour her hat is?]

(iii) The King now changes some of the hats, and then says "Each of your hats is either black or white. At least one of you has a white hat."

He now asks them all "Do you know what colour your hat is?". They all simultaneously reply "No".

What can you deduce about the colour of their hats? Explain your answer.

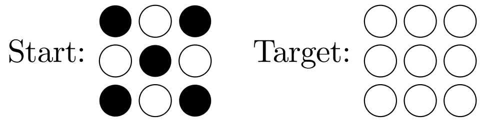
(iv) He again asks "Do you know what colour your hat is?" Alice says "No", but Bob and Charlie both say "Yes" (all three answer simultaneously).

What colour are their hats? Explain your answer.

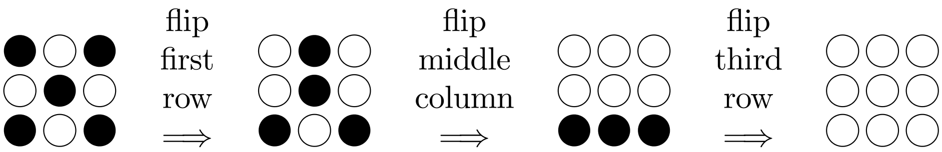
7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

The game of *Oxflip* is for one player and involves circular counters, which are white on one side and black on the other, placed in a grid. During a game, the counters are flipped over (changing between black and white side uppermost) following certain rules.

Given a particular size of grid and a set starting pattern of whites and blacks, the aim of the game is to reach a certain target pattern. Each "move" of the game is to flip over either a whole row or a whole column of counters (so one whole row or column has all its blacks swapped to whites and vice versa). For example, in a game played in a three-by-three square grid, if you are given the starting and target patterns

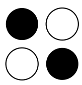


a sequences of three moves to achieve the target is:

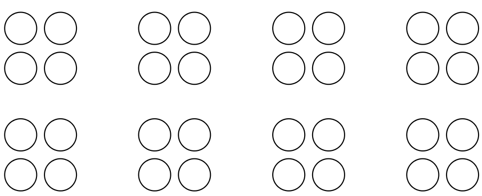


There are many other sequences of moves which also have the same result.

(i) Consider the two-by-two version of the game with starting pattern

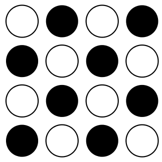


Draw, in the blank patterns below, the eight different target patterns (including the starting pattern) that it is possible to obtain.



What are the possible numbers of white counters that may be present in these target patterns?

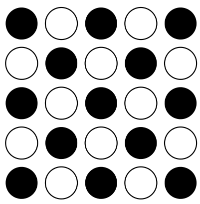
(ii) In the four-by-four version of the game, starting with pattern



explain why it is impossible to reach a pattern with only one white counter.

[*Hint: don't try to write out every possible combination of moves.*]

(iii) In the five-by-five game, explain why any sequence of moves which begins



and ends with an all-white pattern, must involve an odd number of moves. What is the least number of moves needed? Give reasons for your answer.

End of Last Question