Solutions to INDUCTION EXERCISES 1

1. We have, for $0 \leqslant k \leqslant n$,

$$\begin{aligned}
\text{LHS} &= \binom{n}{k} + \binom{n}{k+1} \\
&= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-k-1)!} \\
&= \frac{n!}{(k+1)!\,(n-k)!}\left\{(k+1)+(n-k)\right\} \quad \text{[putting this over a common denominator]} \\
&= \frac{n!\,(n+1)}{(k+1)!\,(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!\,((n+1)-(k+1))!} \\
&= \binom{n+1}{k+1} = \text{RHS}.
\end{aligned}$$

Now let n be a natural number and x, y real numbers. The Binomial Theorem is easy to check for $n = 0$ as we have

$$\text{LHS} = (x+y)^0 = 1, \qquad \text{RHS} = \binom{0}{0} x^0 y^0 = 1.$$

As our inductive hypothesis, we assume that the Binomial Theorem is true for $n = N$. Then

$$\begin{aligned}
\text{LHS} &= (x+y)^{N+1} \\
&= (x+y)\,(x+y)^N \\
&= (x+y)\left(\sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}\right) \quad \text{[writing in our assumed expression for } (x+y)^N\text{]} \\
&= \sum_{k=0}^{N} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=0}^{N} \binom{N}{k} x^k y^{N+1-k} \\
&= x^{N+1} + \sum_{k=0}^{N-1} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=1}^{N} \binom{N}{k} x^k y^{N+1-k} + y^{N+1} \quad \text{[taking out one term from each sum]} \\
&= x^{N+1} + \sum_{l=1}^{N} \binom{N}{l-1} x^l y^{N+1-l} + \sum_{k=1}^{N} \binom{N}{k} x^k y^{N+1-k} + y^{N+1} \quad \text{[setting } l = k+1 \text{ in the first sum]} \\
&= x^{N+1} + \sum_{k=1}^{N} \left\{\binom{N}{k-1} + \binom{N}{k}\right\} x^k y^{N+1-k} + y^{N+1} \\
&= x^{N+1} + \sum_{k=1}^{N} \binom{N+1}{k} x^k y^{N+1-k} + y^{N+1} \quad \text{[using the first part of the question]} \\
&= \sum_{k=0}^{N+1} \binom{N+1}{k} x^k y^{N+1-k} = \text{RHS}.
\end{aligned}$$

2. To prove, by induction, that

$$\sum_{r=1}^{n} \frac{1}{r^2} \leqslant 2 - \frac{1}{n}, \tag{1}$$

we first check that LHS$(n=1) = \sum_{r=1}^{1} \frac{1}{r^2} = 1 \leqslant 2 - \frac{1}{1} = \text{RHS}(n=1)$, and assuming that (1) is correct for $n = N$, we complete the proof by noting

$$\begin{aligned}
\text{LHS}\,(n = N+1) &= \sum_{r=1}^{N+1} \frac{1}{r^2} \\
&= \sum_{r=1}^{N} \frac{1}{r^2} + \frac{1}{(N+1)^2} \\
&\leqslant 2 - \frac{1}{N} + \frac{1}{(N+1)^2} \\
&= 2 - \frac{1}{N+1}\left(\frac{N^2+N+1}{N\,(N+1)}\right) \\
&\leqslant 2 - \frac{1}{N+1}\left(\frac{N^2+N}{N^2+N}\right) \\
&= 2 - \frac{1}{N+1} = \text{RHS}\,(n = N+1)\,.
\end{aligned}$$

3. To prove by induction

$$\sqrt{n} \leqslant \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant 2\sqrt{n} - 1, \tag{2}$$

for $n \geqslant 1$, we first check the $n = 1$ case:

$$\sqrt{1} \leqslant \sum_{k=1}^{1} \frac{1}{\sqrt{k}} \leqslant 2\sqrt{1} - 1,$$

as all three quantities in the inequality equal 1. Assuming that (2) is correct for the case $n = N$, we would have

$$2\sqrt{N} - 1 + \frac{1}{\sqrt{N+1}} \geqslant \sum_{k=1}^{N+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{N} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{N+1}} \geqslant \sqrt{N} + \frac{1}{\sqrt{N+1}}.$$

The desired inequalities for $n = N+1$ are

$$\begin{aligned}
&\sqrt{N} + \frac{1}{\sqrt{N+1}} \geqslant \sqrt{N+1} \\
\Leftrightarrow\quad &\sqrt{N} \geqslant \frac{(N+1)-1}{\sqrt{N+1}} \\
\Leftrightarrow\quad &\sqrt{N} \geqslant \frac{N}{\sqrt{N+1}} \\
\Leftrightarrow\quad &\sqrt{N+1} \geqslant \sqrt{N},
\end{aligned}$$

and

$$\begin{aligned}
&2\sqrt{N} - 1 + \frac{1}{\sqrt{N+1}} \leqslant 2\sqrt{N+1} - 1 \\
\Leftrightarrow\quad &2\sqrt{N} \leqslant \frac{2\,(N+1)-1}{\sqrt{N+1}} \\
\Leftrightarrow\quad &2\sqrt{N\,(N+1)} \leqslant 2N+1 \\
\Leftrightarrow\quad &4N^2 + 4N \leqslant 4N^2 + 4N + 1.
\end{aligned}$$

These both hold for all $N \geqslant 1$ and hence (2) follows by induction, as we've shown

$$2\sqrt{N+1} - 1 \geqslant \sum_{k=1}^{N+1} \frac{1}{\sqrt{k}} \geqslant \sqrt{N+1}.$$

4. Let $A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$. It is easy to check, for $n = 1$, the formula

$$\begin{aligned} A^n &= 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}; \qquad (3) \\ \text{RHS}\,(n=1) &= 3^0 \begin{pmatrix} 2+3 & -1 \\ 4 & 3-2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} = A = \text{LHS}\,(n=1)\,. \end{aligned}$$

If we assume as our inductive hypothesis that (3) is true when $n = N$, then

$$\begin{aligned} \text{LHS}\,(n = N+1) &= A^{N+1} = AA^N \\ &= \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \times 3^{N-1} \begin{pmatrix} 2N+3 & -N \\ 4N & 3-2N \end{pmatrix} \quad \text{[by assumption]} \\ &= 3^{N-1} \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2N+3 & -N \\ 4N & 3-2N \end{pmatrix} \\ &= 3^{N-1} \begin{pmatrix} 10N+15-4N & -5N-3+2N \\ 8N+12+4N & -4N+3-2N \end{pmatrix} \\ &= 3^{N-1} \begin{pmatrix} 6N+15 & -3N-3 \\ 12N+12 & -6N+3 \end{pmatrix} \\ &= 3^{N} \begin{pmatrix} 2N+5 & -N-1 \\ 4N+4 & -2N+1 \end{pmatrix} \\ &= 3^{(N+1)-1} \begin{pmatrix} 2(N+1)+3 & -(N+1) \\ 4(N+1) & 3-2(N+1) \end{pmatrix} = \text{RHS}\,(n = N+1)\,, \end{aligned}$$

thus proving (3) to be true for $n \geqslant 1$. In fact, the equation (3) makes some sense for all real numbers n; in particular the expression we get by putting in $n = 1/2$ (as we're looking for a "square root" of A) gives a matrix

$$B = \frac{1}{\sqrt{3}} \begin{pmatrix} 4 & -1/2 \\ 2 & 2 \end{pmatrix}$$

which satisfies

$$B^2 = \frac{1}{3} \begin{pmatrix} 16-1 & -2-1 \\ 8+4 & -1+4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 15 & -3 \\ 12 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} = A.$$

More generally, if we set

$$A\,(x) = 3^{x-1} \begin{pmatrix} 2x+3 & -x \\ 4x & 3-2x \end{pmatrix}$$

for any real number x then it is easy to check that $A\,(x)\,A\,(y) = A\,(x+y)$.

5. k is a positive, and for now, *fixed* integer. We will prove by induction on n the statement at

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)}{k+1} \qquad (4)$$

for $n \geqslant 1$. The initial $n = 1$ case of (4) reduces to LHS $= \sum_{r=1}^{1} r = 1 = \frac{1\times 2}{2} =$ RHS, which is clearly correct. Now suppose that (4) is true for the case $n = N$; then

$$\begin{aligned} \text{LHS}\,(n = N+1) &= \sum_{r=1}^{N+1} r(r+1)(r+2)\cdots(r+k-1) \\ &= \sum_{r=1}^{N} r(r+1)(r+2)\cdots(r+k-1) + (N+1)\,(N+2)\cdots(N+k) \\ &= \frac{N(N+1)(N+2)\cdots(N+k)}{k+1} + (N+1)\,(N+2)\cdots(N+k) \quad \text{[by assumption]} \\ &= \frac{(N+1)\,(N+2)\cdots(N+k)}{k+1} \{N + (k+1)\} \\ &= \frac{(N+1)\,(N+2)\cdots(N+1+k)}{k+1} = \text{RHS}\,(n = N+1)\,. \end{aligned}$$

By induction the first part follows.

Now we will vary k to prove by induction the formula (4). We will need to treat all n *simultaneously* as we proceed through all the values of k — we will also need to use the strong form of induction. So our Kth inductive hypothesis will be that

$$\sum_{r=1}^{n} r^k = \frac{n^{k+1}}{k+1} + E_k(n) \tag{5}$$

is true for all k less than or equal to a particular value K and for all values of $n \geqslant 1$. The case $k = 1$ is easily checked as we have

$$\text{LHS}\,(k=1) = \sum_{r=1}^{n} r = \frac{n\,(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} = \text{RHS}\,(k=1)$$

with $E_1(n) = n/2$, which is a well-known formula from the study of arithmetic progressions, and we know that this formula holds for all $n \geqslant 1$. Now suppose that (5) is true for $k \leqslant K$ and for all $n \geqslant 1$. In particular this means that for any $k \leqslant K$ there is a polynomial Q_{k+1} of degree less $k+1$ such that

$$\sum_{r=1}^{n} r^k = Q_{k+1}(n) \quad \text{for all } n \geqslant 1. \tag{6}$$

From (4) we have

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+K) = \frac{n(n+1)(n+2)\cdots(n+K+1)}{K+2} \quad \text{for all } n \geqslant 1,$$

and hence expanding the brackets

$$\sum_{r=1}^{n} \{r^{K+1} + \text{a polynomial in } r \text{ of order } K\} = \frac{n^{K+2}}{K+2} + \text{another polynomial in } n \text{ of order } \leqslant K+1 \quad \text{for all } n \geqslant 1.$$

From (6) we then have that

$$\left\{\sum_{r=1}^{n} r^{K+1}\right\} + \text{a polynomial in } n \text{ of order } \leqslant K+1 = \frac{n^{K+2}}{K+2} + \text{a polynomial in } n \text{ of order } \leqslant K+1 \quad \text{for all } n \geqslant 1,$$

and simplifies to

$$\sum_{r=1}^{n} r^{K+1} = \frac{n^{K+2}}{K+2} + \text{a polynomial in } n \text{ of order } \leqslant K+1 \quad \text{for all } n \geqslant 1,$$

or equivalently to

$$\sum_{r=1}^{n} r^{K+1} = \frac{n^{K+2}}{K+2} + E_{K+1}(n) \quad \text{for all } n \geqslant 1,$$

if we call this polynomial E_{K+1}. Hence (5) follows by induction on k for all $k \geqslant 1$ and $n \geqslant 1$.

Alternative Method: Here is another way to prove equation (5) in one stage. Suppose as our inductive hypothesis that there are polynomial E_k for $k \leqslant K$ such that (5) holds for all $n \geqslant 1$. Then we can note that

$$\sum_{r=1}^{n} \left\{(r+1)^{K+1} - r^{K+1}\right\} = \sum_{r=2}^{n+1} r^{K+1} - \sum_{r=1}^{n} r^{K+1} = (n+1)^{K+1} - 1$$

as all but two of the terms in the two sums cancel. Now expanding the brackets in the above equation with the binomial theorem we find

$$\sum_{r=1}^{n} \{r^{K+1} + (K+1)\,r^K + (\text{polynomial in } r \text{ of order } K-1) - r^{K+1}\} = n^{K+1} + \text{polynomial in } n \text{ of order } K.$$

Cancelling terms and using our inductive hypothesis we see

$$\left\{(K+1)\sum_{r=1}^{n} r^K\right\} + (\text{polynomial in } n \text{ of order } K) = n^{K+1} + \text{polynomial in } n \text{ of order } K.$$

Hence

$$\sum_{r=1}^{n} r^K = \frac{n^{K+1}}{K+1} + E_{K+1}(n)\,,$$

for some polynomial E_{K+1}, and hence the result follows.

Solutions to INDUCTION EXERCISES 2

1. As our inductive hypothesis, we shall assume that n lines in the plane, such that:

- no two of which are parallel,
- no three of which meet in a common point,

divide the plane into $\frac{1}{2}(n^2+n+2)$ regions. This hypothesis is clearly true for the initial $n=0$ case, where the formula gives 1 region – the undivided plane.

Consider the addition of an $(N+1)$th new line, to N lines already in place. How many new regions will be created? This new line will intersect each of the other N lines, as it is parallel to none of them. Further, the N intersections of this new line with the other lines are N distinct points along the new line, as no three lines are concurrent. These N distinct intersections on the new line divide it into $N+1$ line segments. Each of these line segments divides, into two regions, what previously had been only one region. That is, the addition of the new line has created $N+1$ new regions.

If our inductive hypothesis held true for $n=N$, then we now have, in total,

$$\frac{1}{2}(N^2+N+2)+(N+1)=\frac{1}{2}(N^2+3N+4)=\frac{1}{2}\{(N+1)^2+(N+1)+1\}$$

regions. This is the correct formula for $n=N+1$, verifying the inductive hypothesis in this case.

2. We will write $A(n)=3^{3n-2}+2^{3n+1}$ for $n\geqslant 1$. It is easy to check that

$$A(1)=3^1+2^4=3+16=19$$

is divisible by 19.

As an inductive hypothesis, we will assume that $A(N)$ is divisible by 19. If we compare the expression for $A(N+1)$ with that of $A(N)$, we see that part of the formula has increased by $3^3=27$, the other part increasing by $2^3=8$. If we consider either

$$A(N+1)-27A(N) \text{ or } A(N+1)-8A(N),$$

we will simplify the expression by eliminating either the powers of 3, or the powers of 2. So

$$\begin{aligned} A(N+1)-27A(N) &= (3^{3N+1}+2^{3N+4})-27\times(3^{3N-2}+2^{3N+1}) \\ &= 2^{3N+4}-27\times 2^{3N+1} \\ &= (8-27)\times 2^{3N+1} \\ &= -19\times 2^{3N+1}. \end{aligned}$$

The RHS is clearly divisible by 19; by hypothesis $A(N)$, and hence $27A(N)$, is also divisible by 19. Thus $A(N+1)$ is also divisible by 19, verifying the inductive hypothesis for $n=N+1$.

3. (a) As $u^2 - 2v^2 = 1$ then

$$\text{LHS} = (3u+4v)^2 - 2(2u+3v)^2 = (9u^2+24uv+16v^2) - (8u^2+24uv+18v^2) = u^2 - 2v^2 = 1 = \text{RHS}.$$

(b) If u_n and v_n satisfy $(u_n)^2 - 2(v_n)^2 = 1$ then

$$u_{n+1} = 3u_n + 4v_n \text{ and } v_{n+1} = 2u_n + 3v_n$$

also satisfy this equation $u^2 - 2v^2 = 1$. So beginning with $u_0 = 3$ and $v_0 = 2$, noting that $(u_0)^2 - 2(v_0)^2 = 9 - 2 \times 4 = 1$, we can generate recursively infinitely many solutions (u_n, v_n) for $n \geqslant 1$.

(c) Any pair (u, v), which satisfies $u^2 - 2v^2 = 1$, also satisfies

$$\frac{u}{v} = \sqrt{2 + \frac{1}{v^2}},$$

and so u/v is a good rational approximation of $\sqrt{2}$. Moreover, the greater v is the better u/v is as an approximation. As

$$v_{n+1} = 2u_n + 3v_n > v_n$$

then u_{n+1}/v_{n+1} is a better approximation to $\sqrt{2}$ than u_n/v_n.

If we begin with $u_0 = 3$ and $v_0 = 2$ then we generate the following rational approximations to $\sqrt{2} = 1.414213562\ldots$

n	u_n	v_n	u_n/v_n
0	3	2	1.5000000
1	17	12	1.4166667
2	99	70	1.4142857
3	577	408	1.4142157
4	3363	2378	1.4142136

and so we see that by the fourth approximation u_n/v_n is accurate to 6 decimal places.

4. Suppose, as our inductive hypothesis that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \tag{7}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. As our expression for F_n is in terms of F_{n-2} and F_{n-1}, we will need to know equation (7) holds for *two* consecutive Fibonacci numbers in order to be able to deduce anything about the next Fibonacci number. Checking the $n = 0$ and $n = 1$ cases we see

$$\begin{aligned} \text{when } n &= 0\text{: LHS of equation (7)} = F_0 = 0 = \frac{1-1}{\sqrt{5}} = \frac{\alpha^0 - \beta^0}{\sqrt{5}} = \text{RHS, and} \\ \text{when } n &= 1\text{: LHS of equation (7)} = F_1 = 1 = \frac{\sqrt{5}}{\sqrt{5}} = \frac{\alpha - \beta}{\sqrt{5}} = \text{RHS.} \end{aligned}$$

Given our inductive hypothesis (7) holds for $n = N-1$ and $n = N$, and the defining relation of the Fibonacci numbers, we may write

$$F_{N+1} = F_N + F_{N-1} = \frac{\alpha^N - \beta^N}{\sqrt{5}} + \frac{\alpha^{N-1} - \beta^{N-1}}{\sqrt{5}} = \frac{\alpha^{N-1}(\alpha+1)}{\sqrt{5}} - \frac{\beta^{N-1}(\beta+1)}{\sqrt{5}}.$$

At this point we note that α and β are the two roots of the quadratic $1 + x = x^2$ (check this!) giving

$$F_{N+1} = \frac{\alpha^{N-1}\alpha^2}{\sqrt{5}} - \frac{\beta^{N-1}\beta^2}{\sqrt{5}} = \frac{\alpha^{N+1} - \beta^{N+1}}{\sqrt{5}}$$

which is the correct form, showing our assumption (7) holds when $n = N+1$. We now have two new consecutive numbers, N and $N+1$, for which (7) is true, which allow us to prove the $N+2$ case; knowledge of the $N+1$ and $N+2$ cases demonstrates the $N+3$ case etc. etc. and the result follows by induction.

5. (a) The sequence of numbers $x_0, x_1, x_2, \ldots$ begins with $x_0 = 1$ and $x_1 = 1$ and is then is recursively determined by the equations

$$x_{n+2} = 4x_{n+1} - 3x_n + 3^n \quad \text{for} \quad n \geqslant 0,$$

so that

$$\begin{aligned} x_2 &= 4x_1 - 3x_0 + 3^0 = 4 - 3 + 1 = 2; \\ x_3 &= 4x_2 - 3x_1 + 3^1 = 8 - 3 + 3 = 8; \\ x_4 &= 4x_3 - 3x_2 + 3^2 = 32 - 6 + 9 = 35; \\ x_5 &= 4x_4 - 3x_3 + 3^3 = 140 - 24 + 27 = 143. \end{aligned}$$

(b) There are unique values A, B, C such that $x_n = A + B \times 3^n + C \times n3^n$ agrees with x_0, x_1, x_2. We can find these by substituting this expression into the equations:

$$x_0 = 1 = A + B, \tag{8}$$

$$x_1 = 1 = A + 3B + 3C, \tag{9}$$

$$x_2 = 2 = A + 9B + 18C. \tag{10}$$

Subtracting (8) from (9) and (10) gives

$$0 = 2B + 3C \quad \text{and} \quad 1 = 8B + 18C$$

which give

$$B = \frac{-1}{4} \quad \text{and} \quad C = \frac{1}{6} \Rightarrow A = \frac{5}{4}.$$

Hence the only possible A, B, C values to agree with the cases $n = 0, 1, 2$, gives the formula

$$x_n = \frac{5 - 3^n}{4} + \frac{n3^n}{6}. \tag{11}$$

We also note that

$$\begin{aligned} x_3 &= \frac{5 - 27}{4} + \frac{81}{6} = \frac{-11}{2} + \frac{27}{2} = 8 \\ x_4 &= \frac{5 - 81}{4} + \frac{324}{6} = -19 + 54 = 35 \\ x_5 &= \frac{5 - 243}{4} + \frac{1215}{6} = \frac{-119}{2} + \frac{405}{2} = 143, \end{aligned}$$

and so the expression agrees with the cases $n = 3, 4, 5$ also.

(c) Suppose now (for the sake of induction) that the expression (11) is correct for $n = N$ and $n = N + 1$. Then

$$\begin{aligned} x_{N+2} &= 4x_{N+1} - 3x_N + 3^N \\ &= 4\left(\frac{5 - 3^{N+1}}{4} + \frac{(N+1)\,3^{N+1}}{6}\right) - 3\left(\frac{5 - 3^N}{4} + \frac{N3^N}{6}\right) + 3^N \\ &= \frac{20 - 15}{4} + 3^N\left(-3 + (2N+2) + \frac{3}{4} - \frac{N}{2} + 1\right) \\ &= \frac{5}{4} + 3^N\left(\frac{6N+3}{4}\right) \\ &= \frac{5}{4} + 3^N\left(\frac{-3 + 6\,(N+2)}{4}\right) \\ &= \frac{5 - 3^{N+1}}{4} + \frac{(N+2)\,3^{N+2}}{6}, \end{aligned}$$

thus proving (11) is correct for all $n \geqslant 0$.

Solutions to ALGEBRA EXERCISES 1

1. (a) The remainders when n^2+4 is divided by 7 for each of $0 \leqslant n < 7$ are

n	0	1	2	3	4	5	6
n^2+4	4	5	8	13	20	29	40
Remainder	4	5	1	6	6	1	5

So clearly n^2+4 is not divisible by 7 for any of $0 \leqslant n < 7$.

Now any integer n can be written in the form $n = 7k + r$ where k is an integer and r is in the range $0 \leqslant r < 7$. (So that k is the number of times 7 'goes into' n and r is the remainder.) Note for this n that

$$\begin{aligned} n^2+4 &= (7k+r)^2+4 \\ &= 49k^2+14kr+r^2+4 \\ &= 7(7k+2r)+r^2+4. \end{aligned}$$

So the remainder when n^2+4 is divided by 7 is the same as when r^2+4 is divided by 7, because n^2+4 and r^2+4 differ by a multiple of 7. From the above we know that r^2+4 is never a multiple of 7 and hence neither is n^2+4.

(b) Similarly, any integer n can be written as $4i+r$ where $0 \leqslant r < 4$. Note that

$$\begin{aligned} n^3+k &= (4i+r)^3+k \\ &= 64i^3+48i^2r+16ir^2+r^3+k \\ &= 4\left(16i^3+12i^2r+4ir^2\right)+r^3+k. \end{aligned}$$

So n^3+k and r^3+k leave the same remainder when divided by 4. So our problem simplifies to asking:

"For what values of k is none of $k+0$, $k+1$, $k+8$, $k+27$ divisible by 4?"

Recall that k itself can be written in one of the forms $4l+m$, where l is an integer and $m = 0, 1, 2, 3$. Note

$$\begin{aligned} \text{if } k+0 &= 4l+m \text{ is not divisible by 4 then } m \neq 0; \\ \text{if } k+1 &= 4(l+1)+(m-3) \text{ is not divisible by 4 then } m \neq 3; \\ \text{if } k+8 &= 4(l+2)+m \text{ is not divisible by 4 then } m \neq 0; \\ \text{if } k+27 &= 4(k+7)+(m-1) \text{ is not divisible by 4 then } m \neq 1. \end{aligned}$$

So $m = 2$ is the only remaining possibility, and k must be of the form $4l+2$ — that is k is on the list $\pm 2, \pm 6, \pm 10, \pm 14, \ldots$

2. (i) Let a and b be positive real numbers. Then

$$\begin{aligned} \sqrt{ab} &\leqslant \frac{1}{2}(a+b) \qquad (12) \\ \Leftrightarrow ab &\leqslant \frac{1}{4}(a+b)^2 \\ \Leftrightarrow 4ab &\leqslant a^2+2ab+b^2 \\ \Leftrightarrow 0 &\leqslant a^2-2ab+b^2 \\ &\Leftrightarrow 0 \leqslant (a-b)^2, \end{aligned}$$

which is always true as the square of a real number is non-negative. Note that we have equality in these inequalities when a and b are equal.

(ii) Now let a_1, a_2, ..., a_n be positive real numbers, with sum $S = a_1 + a_2 + \cdots + a_n$ and product $P = a_1 a_2 \cdots a_n$, and suppose that two of these numbers, a_i and a_j say, are distinct. If we replace a_i and a_j with $(a_i + a_j)/2$ and $(a_i + a_j)/2$ then we create a new sum S' and a new product P'. Note that the contribution to the new sum S' from the two new elements equals

$$\left(\frac{a_i + a_j}{2}\right) + \left(\frac{a_i + a_j}{2}\right) = a_i + a_j$$

which is the same as the contribution to the previous sum, whilst the contribution to the new product P' is

$$\left(\frac{a_i + a_j}{2}\right)^2 \text{ which strictly exceeds } a_i a_j$$

by (12). So the contribution to the new product P' from the two new elements is strictly greater than the contribution $a_i a_j$ to the original product P. This means, that if any of the numbers are distinct, we can improve on the product without changing the sum.

So we know the largest possible product, that can be made from n numbers which add up to S, is achieved when all the numbers are equal; that is they each equal S/n. So for our original n numbers $a_1, a_2, \ldots, a_n$, which might be distinct, we have

$$a_1 a_2 \cdots a_n \leqslant \left(\frac{S}{n}\right)^n = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$$

and taking nth roots of both sides gives

$$(a_1 a_2 \cdots a_n)^{1/n} \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality only when the a_i are all equal.

[The quanities $(a_1 a_2 \cdots a_n)^{1/n}$ and $(a_1 + a_2 + \cdots + a_n)/n$ are respectively called the *geometric mean* and *arithmetic mean* of $a_1, a_2, \ldots, a_n$.]

3. (i) Let n be a positive integer. One method of approach is to note that

$$x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$$

is a geometric progression with n terms, first term x^{n-1} and common ration y/x. So it sums to

$$\frac{x^{n-1}\{1 - (y/x)^n\}}{1 - y/x} = \frac{x^n - y^n}{x - y}.$$

(The identity is trivial when $x = 0$.) Alternatively, one can simply multiply out the RHS brackets.

(ii) Let a be a positive integer. From the previous part we see that $a^n - 1$ is divisible by $a - 1$. In the case that $a^n - 1$ is prime it must follow that $a - 1 = 1$, that is $a = 2$.

Further, if n is not prime and may be written $n = rs$ where $r, s > 1$ then using the first part once more we note that $2^n - 1 = (2^r)^s - 1$ is divisible by $2^r - 1$; that is, for $2^n - 1$ to be prime, it must also follow that n is prime.

The converse, that $2^n - 1$ is prime when n is prime, is not true. For example, when $n = 11$ we have

$$2^{11} - 1 = 2047 = 23 \times 89.$$

Primes of the form $2^n - 1$ are called *Mersenne* primes. Whether or not there are infinitely many Mersenne primes is still an open problem.

4. Let a, b, r, s be rational numbers, such that $r + s\sqrt{2}$ is a root of

$$x^2 + ax + b = 0. \tag{13}$$

Substituting in $x = r + s\sqrt{2}$ into (13) gives

$$(r + s\sqrt{2})^2 + a(r + s\sqrt{2}) + b = (r^2 + 2s^2 + ar + b) + (2rs + as)\sqrt{2} = 0.$$

Note that $r^2 + 2s^2 + ar + b$ and $2rs + as$ are rational numbers still. Hence, as $\sqrt{2}$ is irrational, it must be the case that

$$r^2 + 2s^2 + ar + b = 0 = 2rs + as.$$

In which case, substituting $x = r - s\sqrt{2}$ into (13) gives

$$(r - s\sqrt{2})^2 + a(r - s\sqrt{2}) + b = (r^2 + 2s^2 + ar + b) - (2rs + as)\sqrt{2} = 0,$$

and so $r - s\sqrt{2}$ is also a root of (13).

5. (i) As the cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ, then

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(x-\alpha)(x-\beta)(x-\gamma) \\ &= a\left(x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\right). \end{aligned}$$

Comparing the coefficients of x^2, x^1, x^0 respectively, we see

$$\alpha + \beta + \gamma = -\frac{b}{a}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \qquad \alpha\beta\gamma = -\frac{d}{a}. \tag{14}$$

(ii) Recalling $\cos(A + B) = \cos A \cos B - \sin A \sin B$, and the $\cos(2\theta), \sin(2\theta)$ formulas, we have

$$\begin{aligned} \cos 3\theta &= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

(iii) We can check easily that the three roots of

$$4x^3 - 3x - \cos 3\theta = 0 \tag{15}$$

are $\cos\theta, \cos(\theta + 2\pi/3)$ and $\cos(\theta + 4\pi/3)$, by substituting these into the equation — this is clear for $x = \cos\theta$, but also note when $x = \cos(\theta + 2\pi/3)$ that

$$\begin{aligned} \text{LHS of (15)} &= 4\cos^3(\theta + 2\pi/3) - 3\cos(\theta + 2\pi/3) - \cos 3\theta \\ &= \cos(3(\theta + 2\pi/3)) - \cos 3\theta \\ &= \cos(3\theta + 2\pi) - \cos 3\theta = 0, \end{aligned}$$

and a similar calculation shows $x = \cos(\theta + 4\pi/3)$ to be a root. In (15) $d = -\cos 3\theta$ and $a = 4$ and so

$$\cos\theta\cos(\theta + 2\pi/3)\cos(\theta + 4\pi/3) = \frac{\cos(3\theta)}{4}.$$

Note also that in (15), $b = 0$. We see from (14) that the sum of the roots equals $-b/a$ and hence

$$\cos\theta + \cos(\theta + 2\pi/3) + \cos(\theta + 4\pi/3) = 0.$$

Solutions to ALGEBRA EXERCISES 2

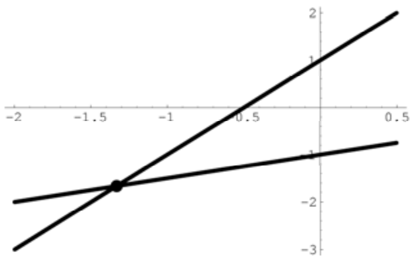
1. The equations

$$ax + by = e \quad \text{and} \quad cx + dy = f \tag{16}$$

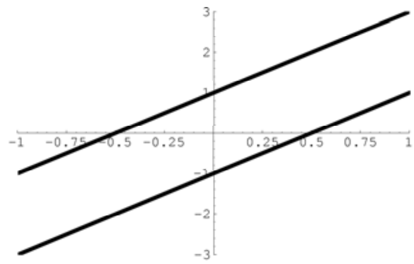
represent two lines in the xy-plane with gradients $-a/b$ and $-c/d$ (including the possibilities that either of these gradients might be infinite if b or d is zero). These lines

- will meet uniquely if they are not parallel; this means their gradients $-a/b$ and $-c/d$ are not equal, or rearranging $ad \neq bc$.
- will never meet if they are parallel and distinct lines; this time the gradients are equal, but the equations are not simply multiples of one another, i,e $ad = bc$, and $af \neq ce$.
- will meet infinitely often if the two equations are multiples of one another and so represent the same line, i.e $ad = bc$ and $af = ce$.

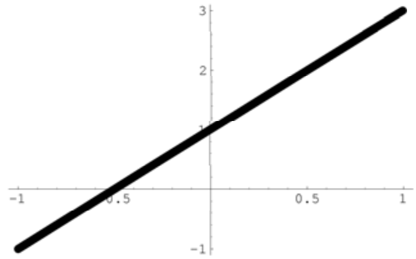
These three possibilities are shown in the diagrams below:



$a/c \neq b/d$: unique solution



$a/c = b/d \neq e/f$: no solutions



$a/c = b/d = e/f$: ∞ solutions

Alternatively, for those with knowledge of matrices, we can look at this from an algebraic (rather than a geometric) point of view. The two equations (16) can be put into one *vector* equation by writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}. \tag{17}$$

- If the 2×2 matrix above is invertible then we can solve (17) to get the unique solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} e \\ f \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.$$

  We know this matrix to be invertible precisely when it has non-zero determinant $ad - bc \neq 0$.

- If the 2×2 matrix has zero determinant, i.e. $ad - bc = 0$, then the column rank of the matrix will be 1 (or 0 if each of a, b, c, d is zero). In this case, one of two things can happen: either the vector $(e \;\; f)^T$ is in the column space of this matrix, in which case the equation is consistent and (17) only really represents one scalar equation repeated twice; or $(e \;\; f)^T$ is not in the column space of the matrix and (17) is inconsistent and has no solutions.

2. From first principles we can aim to directly solve the three simultaneous equations

$$\begin{aligned} x + 2y + a^2 z &= 0; \\ x + ay + z &= 0; \\ x + ay + a^2 z &= 0. \end{aligned}$$

Substituting for x from the first equation into the second two we get:

$$\begin{aligned} (a-2)y + (1-a^2)z &= 0; \\ (a-2)y &= 0. \end{aligned}$$

From question **1**, we know that these equations will have non-zero solutions if they are dependent (i.e they are multiples of one another). This happens when

$$(a-2)(a^2-1) = 0.$$

So the three values of a which lead to non-zero solutions are $a = -1, 1, 2$.

- When $a = -1$ we see that the second and third equations are the same, so that the system effectively reduces to two equations. Setting $z = t$, a parameter in terms of which we will give the general solution, we have:

  $$x + 2y = -t \quad \text{and} \quad x - y = -t.$$

  Solving these gives the general solution

  $$x = t, \quad y = 0, \quad z = t.$$

- When $a = 1$ we see once more that the second and third equations are the same equations. Setting $z = t$, we find:

  $$x + 2y = -t \quad \text{and} \quad x + y = -t.$$

  Solving these gives the same general solution

  $$x = t, \quad y = 0, \quad z = t.$$

- When $a = 2$ we see that the first and third equations are the same. From these

  $$x + 2y + 4z = 0, \qquad x + 2y + z = 0,$$

  we can see that $z = 0$. So instead of z, we can use y as a parameter, and the general solution has the form

  $$x = -2t, \quad y = t, \quad z = 0.$$

For those with knowledge of matrices and determinants, these values of a could be found by setting the 3×3 determinant

$$\det \begin{pmatrix} 1 & 2 & a^2 \\ 1 & a & 1 \\ 1 & a & a^2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & a^2 \\ 0 & a-2 & 1-a^2 \\ 0 & a-2 & 0 \end{pmatrix} = (a-2)(a^2-1)$$

to zero.

3. (i) The matrix

$$\begin{pmatrix} \cos(2\pi/5) & -\sin(2\pi/5) \\ \sin(2\pi/5) & \cos(2\pi/5) \end{pmatrix}$$

represents rotation about the origin anti-clockwise, by an angle of $2\pi/5$ radians. The matrices

$$A, A^2, A^3, A^4, A^5$$

represent anti-clockwise rotations about the origin by $2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5, 2\pi$ respectively.

The last of these is equivalent to the identity, and so $A^5 = I$, whilst $A^i \neq I$ for $1 \leqslant i \leqslant 4$ as these matrices represent non-trivial rotations.

(ii) If we set $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, for example, then $A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix} \neq I$ for all positive integers n.

(iii) Set

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix};$$

then

$$\begin{aligned} AB &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \\ BA &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and so $AB \neq BA$.

(iv) Suppose that A and B are such that AB is invertible and BA is singular. Then

$$\begin{aligned} 0 &\neq \det(AB) = \det A \det B, \\ 0 &= \det(BA) = \det B \det A, \end{aligned}$$

which gives us the required contradiction, and hence no such A and B exist.

(v) Suppose that A satisfies $A^5 = I$ and $A^{11} = 0$. Then

$$0 = A^{11} = \left(A^5\right)^2 A = A$$

and so $A = 0$, which means $A^{11} = 0 \neq I$: a contradiction which shows that no such A can exist.

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix such that $\det A = 1$ and $A^T A = I$. These translate as

$$ad - bc = 1 \quad \text{and} \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From the first column, first row entry of $A^T A$ we can see that $a^2 + c^2 = 1$. Thus (a, c) is a point lying on the unit circle with centre the origin, and so can be written in the form $a = \cos\theta, c = \sin\theta$. Substituting these values back into other equations

$$ab + cd = 0, \qquad ad - bc = 1, \qquad c^2 + d^2 = 1$$

we find

$$b\cos\theta + d\sin\theta, \qquad -b\sin\theta + d\cos\theta = 1, \qquad c^2 + d^2 = 1,$$

which have the unique solution $b = -\sin\theta$ and $d = \cos\theta$. Hence

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This matrix represents an anti-clockwise rotation about the origin by θ. Geometrically the condition $\det A = 1$ is equivalent to this matrix being *area-preserving* and *sense-preserving*. The condition $A^T A = I$ is equivalent to the matrix preserving lengths and angles.

5. (a) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

so that

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}.$$

Then

$$\begin{aligned} \det(AB) &= (a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\delta)(c\alpha + d\gamma) \\ &= (ac\alpha\beta + bc\beta\gamma + ad\alpha\delta + bd\gamma\delta) - (ac\alpha\beta + bc\alpha\delta + ad\beta\gamma + bd\gamma\delta) \\ &= bc\beta\gamma + ad\alpha\delta - bc\alpha\delta - ad\beta\gamma \\ &= (ad - bc)(\alpha\delta - \beta\gamma) \\ &= \det A \det B \end{aligned}$$

as required.

(b) For the above A, we see that $A^2 - (\text{trace}A)A + (\det A)I$ equals:

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - (a + d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (a^2 + bc) - (a + d)a + (ad - bc) & (ab + bd) - (a + d)b \\ (ac + cd) - (a + d)c & (bc + d^2) - (a + d)d + (ad - bc) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This equation verifies the *Cayley-Hamilton Theorem* for 2×2 matrices. This is a second year result from linear algebra, which states that a matrix satisfies its characteristic equation.

(c) Suppose now that $A^n = 0$ for some $n \geqslant 2$. From part (a) we have

$$(\det A)^n = \det(A^n) = \det(0) = 0$$

and so $\det A = 0$. Substituting this into part (b), we find

$$A^2 = (\text{trace}A)A. \tag{18}$$

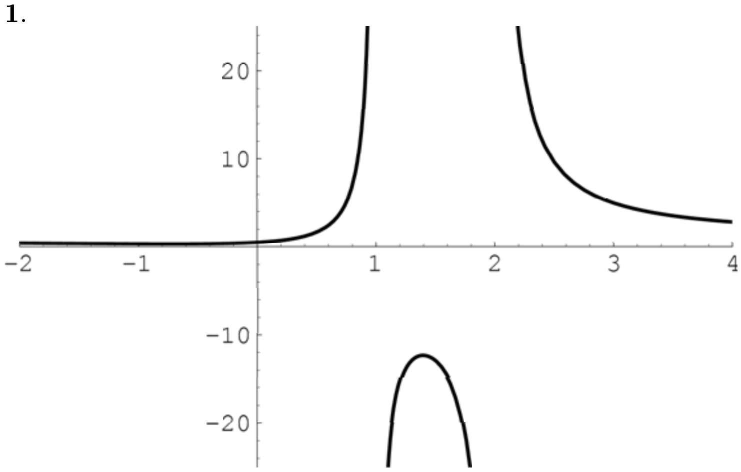
Hence

$$\begin{aligned} 0 &= A^n \\ &= A^{n-2} \times A^2 \\ &= A^{n-2} \times (\text{trace}A)A \\ &= (\text{trace}A)A^{n-1} \\ &= \cdots [\text{repeated substitutions}] \\ &= (\text{trace}A)^{n-2}A^2. \end{aligned}$$

From this we see that either $A^2 = 0$, in which case we have the required result, or $\text{trace}A = 0$. If the latter is the case we see that $A^2 = 0$ again from (18).

Solutions to CALCULUS EXERCISES 1 – Curve Sketching

1.



The graph of

$$y = \frac{x^2+1}{(x-1)(x-2)}$$

for $-2 \leqslant x \leqslant 4$.

By the quotient rule $\mathrm{d}y/\mathrm{d}x$ equals

$$\frac{(x^2-3x+2)(2x)-(x^2+1)(2x-3)}{(x-1)^2(x-2)^2}$$

$$= \frac{-3x^2+2x+3}{(x-1)^2(x-2)^2}.$$

The graph has asymptotes at $x=1$ and $x=2$ with

$$y \to +\infty \text{ as } x \nearrow 1; \qquad y \to -\infty \text{ as } x \searrow 1; \qquad y \to -\infty \text{ as } x \nearrow 2; \qquad y \to +\infty \text{ as } x \searrow 2.$$

Also $y \to 1$ as $x \to \pm\infty$ as

$$y = \frac{x^2+1}{x^2-3x+2} = 1 + \frac{3x-1}{x^2-3x+2}.$$

To find the turning points of the graph, we set the derivative

$$\frac{-3x^2+2x+3}{(x-1)^2(x-2)^2}.$$

to zero. This occurs when $x = (1 \pm \sqrt{10})/3$.

When $x = (1+\sqrt{10})/3 \cong 1.387$ then $y \cong -12.325$ – this is a maximum.

When $x = (1-\sqrt{10})/3 \cong -0.721$ then $y \cong 0.325$ – this is a minimum.

2. The parabola $x = y^2 + ay + b$ passes through the point $(1,1)$ making right angles there with the curve $y = x^2$.

As it passes through $(1,1)$ then $1 = 1 + a + b$ and so $a + b = 0$.

Also, the gradient of the curve $y = x^2$ at $(1,1)$ is 2. So the gradient of $x = y^2 + ax + b$ at $(1,1)$ must be $-1/2$ as the curves make right-angles. By implicitly differentiating, we find

$$1 = 2y\frac{\mathrm{d}y}{\mathrm{d}x} + a\frac{\mathrm{d}y}{\mathrm{d}x} + b.$$

Substituting in $x = y = 1$ and $\mathrm{d}y/\mathrm{d}x = -1/2$ gives

$$1 = \frac{-1}{2}(a+2) \implies a = -4,$$

and hence $b = -a = 4$.

Alternative Method: If we rotate the curve $y = x^2$ clockwise through a right-angle about the point $(1,1)$, then we see the apex of $y = x^2$ moves from the origin to $(0,2)$. The equation of this new parabola is

$$x = (y-2)^2 = y^2 - 4y + 4$$

and it cuts $y = x^2$ at $(1,1)$ in a right-angle by the nature of its construction. We can simply read off that $a = -4$ and $b = 4$.

3. The curve C is the xy-plane has equation

$$x^2 + xy + y^2 = 1. \tag{19}$$

If we differentiate this implicitly we obtain

$$2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

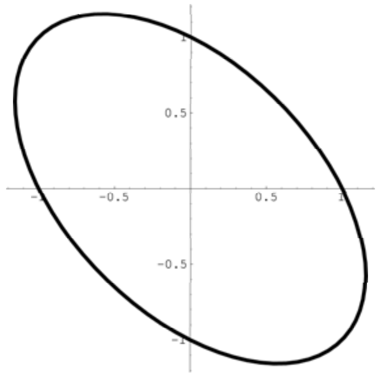
and so $\mathrm{d}y/\mathrm{d}x = 0$ when $y = -2x$. Substituting this back into (19) gives $3x^2 = 1$. So the maximum and minimum values of y are attained when $x = \pm 1/\sqrt{3}$, at which values $y = \mp 2/\sqrt{3}$.

If we change to polar co-ordinates, setting $x = r\cos\theta, y = r\sin\theta$ then the equation (19) becomes

$$r^2(\cos^2\theta + \cos\theta\sin\theta + \sin^2\theta) = 1 \implies r^2 = \left(1 + \frac{1}{2}\sin 2\theta\right)^{-1}.$$

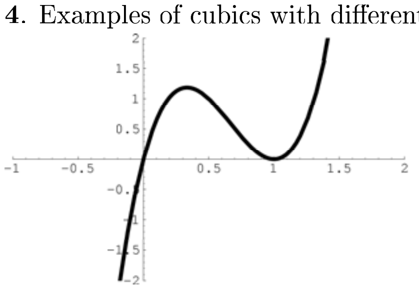
The point of C which is further from the origin is when r is at its largest. This occurs when $\sin 2\theta = -1$, in which case $r = \sqrt{2}$.

A sketch of C is given to the right; the curve is actually an ellipse.

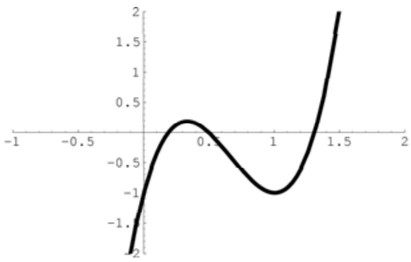


$x^2 + xy + y^2 = 1.$

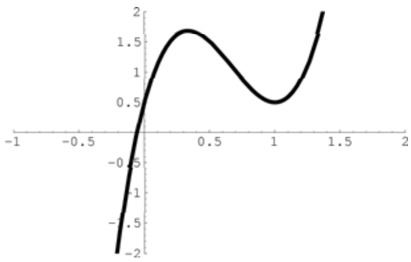
4. Examples of cubics with different numbers of real roots:



Two roots (one repeated)



Three distinct roots



One real root

We are now told that $a < 0$. Differentiating $y = x^3 + ax + b$, and setting the derivative to zero, we see that at the turning points

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + a = 0 \implies x = \pm\sqrt{\frac{-a}{3}}.$$

The cubic $y = x^3 + ax + b$ will have two roots if one of the two turning points lies on $y = 0$, i.e.

$$\begin{aligned} 0 &= \left(\pm\sqrt{\frac{-a}{3}}\right)^3 + a\left(\pm\sqrt{\frac{-a}{3}}\right) + b \\ &= \mp\frac{a}{3}\sqrt{\frac{-a}{3}} \pm a\sqrt{\frac{-a}{3}} + b \\ &= \pm\frac{2a}{3}\sqrt{\frac{-a}{3}} + b, \end{aligned}$$

and so, for a fixed a, the two values of b for which the cubic has two roots are

$$b_- = \frac{2a}{3}\sqrt{\frac{-a}{3}} \quad \text{and} \quad b_+ = \frac{-2a}{3}\sqrt{\frac{-a}{3}}.$$

Now as we vary b (keeping a fixed) this has the effect of translating the graph up and down. As we increase b from above b_- the repeated root becomes two distinct roots the cubic will keep three roots until b becomes b_+ and two of the roots agree. So the cubic has three roots when

$$b_- < b < b_+.$$

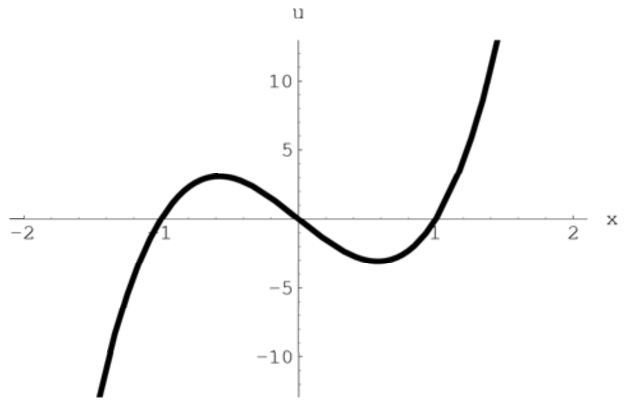
5. To find the turning points of the cubic $u = 8\left(x^3 - x\right)$ we differentiate to see that

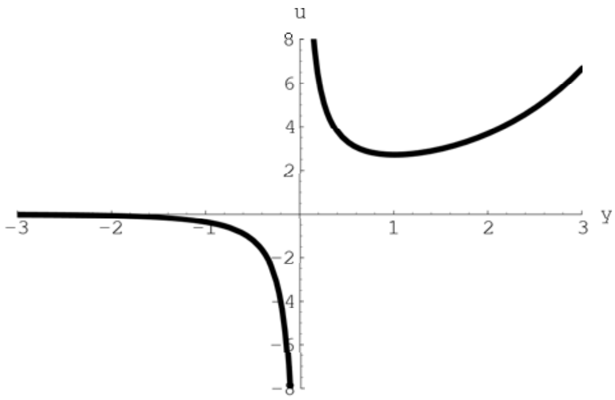
$$\frac{\mathrm{d}u}{\mathrm{d}x} = 8\left(3x^2 - 1\right) = 0 \text{ when } x = \pm\frac{1}{\sqrt{3}} \text{ and } u = \pm\frac{8}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \mp\frac{16}{3\sqrt{3}}.$$

The turning point of the function $u = e^y/y$ can be similarly found: we have

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{-1}{y^2}e^y + \frac{1}{y}e^y = \frac{e^y\left(y-1\right)}{y^2} = 0 \text{ when } y = 1 \text{ and } u = \frac{e^1}{1} = e.$$

The graphs of these two functions are sketched below.





$u = 8(x^3 - x)$ for $-2 \leqslant x \leqslant 2$ $\qquad$ $u = e^y/y$ for $-3 \leqslant y \leqslant 3$

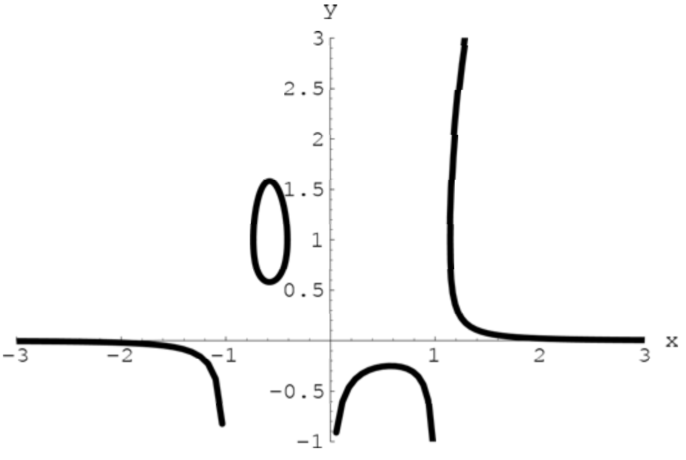
In order to sketch the graph of $e^y = 8y(x^3 - x)$, or equivalently the graph of $e^y/y = 8\left(x^3 - x\right)$, we will let x range from $-\infty$ to ∞, as we do so taking note of the value $u = 8\left(x^3 - x\right)$ from the first graph and seeing, for this u, what values of y (if any) satisfy $u = e^y/y$ from the second graph; we then plot the points (x, y), for the original x and for what solutions y if any we found, on the graph of $e^y = 8y\left(x^3 - x\right)$.

We will see that the graph comes in four stages as we change the value of x . As x varies from $-\infty$ to -1 we see, from the first graph, that $u = 8\left(x^3 - x\right)$ increases from $-\infty$ to 0; for each u in the range $(-\infty, 0)$ we see from the second graph that there is one y which solves $e^y/y = u$. For large negative u this y will be small and negative, and as u becomes closer to 0 then y becomes large and negative.

For x in the range $-1 < x < 0$ we see that u increases from 0 up to $\frac{16}{3\sqrt{3}}$ before decreasing back down to 0. We see that $u = e^y/y$ has no solutions in y for u in the range $0 < u < e$. So initially, as x increases beyond -1 there will be no corresponding solutions in y. However because $\frac{16}{3\sqrt{3}} > e$ then as x continues to increase u moves into a range $(e < u)$ where there are two solutions for y. As x increases beyond $-1/\sqrt{3}$ then u decreases, eventually below e and again we have no solutions in y once more.

For x in the range $0 < x < 1$ we see that u decreases from from 0 down to $\frac{-16}{3\sqrt{3}}$ before increasing back to 0. We see that $u = e^y/y$ has one solution in y for u in this range. From the second graph we see that when u is small and negative the corresponding y will be large and negative, this y will become at its largest for $x = 1/\sqrt{3}$, and then as u increases back to 0 so y will decrease again towards $-\infty$.

Finally for x in the range $1 < x$ then u continues to increase from 0 to ∞. Until u reaches e there will be two solutions in y to $e^y/y = u$, one of which will become small and positive as u becomes larger, the other value of y itself becoming large as u increases. So, putting this all together the graph of $e^y = 8y(x^3 - x)$ looks like:



Solutions to CALCULUS EXERCISES 2 – Numerical Methods and Estimation

1. The principle from calculus, which we will make use of for the first two inequalities, is:

$$\text{if } f(a) = g(a) \text{ and } f'(x) < g'(x) \text{ for } a \leqslant x \leqslant b, \text{ then } f(x) < g(x) \text{ for } a < x \leqslant b.$$

This fact is easily verified by noting for $a < x \leqslant b$ that

$$f(x) = f(a) + \int_a^x f'(t)\ \mathrm{d}t < g(a) + \int_a^x g'(t)\ \mathrm{d}t = g(x).$$

- $\sin\theta < \theta$ for $0 < \theta < \pi/2$: Set $a = 0$, $b = \pi/2$, $f(x) = \sin x$, $g(x) = x$ in the above principle. We need only note that
$$f'(\theta) = \cos\theta < 1 = g'(\theta) \text{ and } f(0) = 0 = g(0).$$
- $\theta < \tan\theta$ for $0 < \theta < \pi/2$: Set $a = 0$, $b = \pi/2$, $f(x) = x$, $g(x) = \tan x$ in the above principle. We need only note that
$$f'(\theta) = 1 < \sec^2\theta = g'(\theta) \text{ and } f(0) = 0 = g(0).$$
- $\cos 2\theta < \cos^2\theta$ for $0 < \theta < \pi/2$: This may be proved using trigonometric identities.
$$\cos 2\theta < \cos^2\theta \iff 2\cos^2\theta - 1 < \cos^2\theta \iff \cos^2\theta < 1,$$
which holds true in the given range.

So if we look to arrange in order the given integrals we can see straight away, for $0 < x < 1 < \pi/2$, that three of the integrands satisfy

$$x^3\cos 2x < x^3\cos^2 x < x^3\cos x.$$

It follows that, ranking the integrals, (d) < (b) < (a). To place integral (c), we note that (a) > (c) holds as $\sin x < x$, and (c) > (b) follows because

$$x^2\sin x\cos x > x^3\cos^2 x \iff \sin x > x\cos x \iff \tan x > x$$

holds in the given range. Hence the ranking of the integrals is (a) > (c) > (b) > (d).

2. Clearly $x = 0$ is a solution of

$$\sin x = \frac{x}{2}. \tag{20}$$

In the range $\pi/2 < x < \pi$, then $\sin x$ is decreasing, whilst $x/2$ is increasing; as

$$\sin\pi/2 = 1 > \pi/4 = (\pi/2)/2 \quad \text{and} \quad \sin\pi = 0 < \pi/2,$$

then there is a unique solution of the equation (20) in the range $\pi/2 < x < \pi$. Arguing along similar lines, or using the symmetry of the equation (20) which is odd, we see there is also a unique solution in the range $-\pi < x < -\pi/2$.

We can show that (20) has no solutions in the range $0 < x < \pi/2$, again using calculus. Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2\cos x}$$

which is negative in the range $0 < x < \pi/2$, because $\tan x > x$ in this range, and so $\sin x/x$ is decreasing in this range. Hence

$$\frac{\sin x}{x} > \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi} > \frac{1}{2} \text{ for } 0 < x < \frac{\pi}{2},$$

showing (20) has no solutions in the range $0 < x < \pi/2$.

Now for $|x| \geqslant \pi$ we have $|x/2| \geqslant \pi/2 > 1$ and so $\sin x = x/2$ has no solutions in the range $|x| \geqslant \pi$.

The roots then of (20) are 0 and $\pm c$ for some $\pi/2 < c < \pi$.

In order to find c we can use the Newton-Raphson iteration. This solves the equation $f(x) = 0$ using the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

from a suitable starting point x_0. Here $f(x) = \sin x - \frac{1}{2}x$ seems an appropriate function and so the iteration is

$$\begin{aligned} x_{n+1} &= x_n - \left(\frac{\sin x_n - \frac{1}{2}x_n}{\cos x_n - \frac{1}{2}} \right) \\ &= x_n - \left(\frac{2\sin x_n - x_n}{2\cos x_n - 1} \right) \\ &= \frac{2x_n \cos x_n - 2\sin x_n}{2\cos x_n - 1} \\ &= 2\left(\frac{x_n - \tan x_n}{2 - \sec x_n} \right) \end{aligned}$$

Beginning with $x_0 = 2$ then we arrive at the values

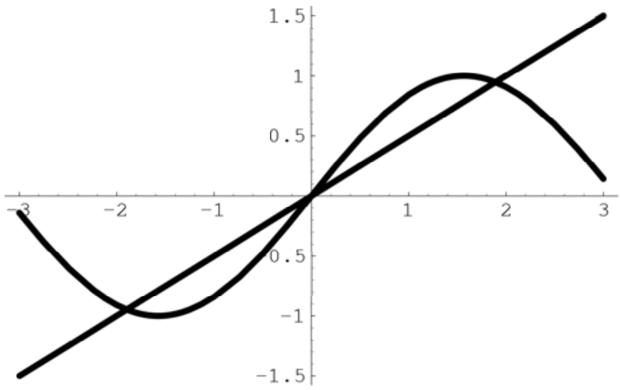
$$x_1 = 1.90100, \quad x_2 = 1.895512, \quad x_3 = 1.895494, \quad x_4 = 1.895494,$$

and so it seems $c = 1.895494$ to 6 d.p. To verify that this is accurate to 6 d.p. we note that

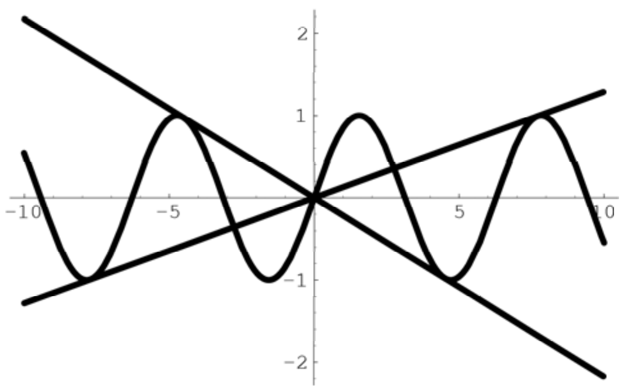
$$f(1.8954945) \cong -1.9 \times 10^{-7} < 0 \text{ and } f(1.8954935) \cong 6.3 \times 10^{-7}$$

and so the root c is in the range $1.8954935 < c < 1.8954945$ and the above answer is correct to 6 d.p.

The equation $\sin x = \lambda x$ has three distinct real roots when $\lambda = \alpha$ or when $\beta < \lambda < 1$ for two real numbers $\alpha < 0 < \beta$, as shown in the second diagram below.



Graphs of $y = \sin x$ and $y = \frac{1}{2}x$



Graphs of $y = \sin x$, $y = \alpha x$, $y = \beta x$

In order to calculate α, though this wasn't asked for in the question, let's label the solutions of $\alpha x = \sin x$ as 0 and $\pm X$. Then, because the graphs of $y = \sin x$ and $y = \alpha x$ touch at $x = X$, that is their y-values and derivatives agree, we have

$$\alpha X = \sin X \quad \text{and} \quad \alpha = \cos X.$$

Hence X is the solution of $x = \tan x$ in the range $\pi < x < 2\pi$; using numerical methods such as Newton-Raphson we can calculate this as $X \cong 4.49341$ and $\alpha = \cos X \cong -0.217233$.

Similarly we can calculate β by looking for the root of $x = \tan x$ in the range $2\pi < x < 3\pi$, and β is then its cosine which comes to approximately 0.128485.

3. (a) A triangle, which connects the origin and two adjacent vertices of I_m, has two equal sides of length 1 and subtends an angle of $2\pi/m$ at the centre. The opposite side then has length $2\sin(\pi/m)$. We then have

$$\text{length of } I_m = 2m\sin\left(\frac{\pi}{m}\right) < 2\pi = \text{ length of } S.$$

Similarly a triangle, which connects the origin and two adjacent vertices of C_n, has height 1, subtends an angle of $2\pi/n$ at the centre, and so has base $2\tan(\pi/n)$. We then have

$$\text{area of } C_n = n\tan\left(\frac{\pi}{n}\right) < \pi = \text{ area of } S.$$

Thus, for $m, n \geqslant 3$,

$$m\sin\left(\frac{\pi}{m}\right) < \pi < n\tan\left(\frac{\pi}{n}\right).$$

(b) To demonstrate Archimedes inequality we need to find m and n such that

$$3\frac{10}{71} < m\sin\left(\frac{\pi}{m}\right) < \pi < n\tan\left(\frac{\pi}{n}\right) < 3\frac{1}{7},$$

and we are asked to find the smallest such m and n.

We could approach the problem of finding the smallest m using trial and error. A more systematic approach is to solve

$$x\sin\left(\frac{\pi}{x}\right) = 3\frac{10}{71} \tag{21}$$

and to take m as the next largest natural number after the solution x. If we write $y = \pi/x$ then equation (21) becomes

$$\sin y = ky \text{ where } k = \frac{3\frac{10}{71}}{\pi}.$$

We have already seen in question **2** how such an equation can be solved using the Newton-Raphson method. In this case we find $y \cong 0.0380098$ and hence

$$x = \frac{\pi}{y} \cong 82.65 \text{ giving the smallest } m = 83.$$

Similarly, to find n, we need to solve

$$x\tan\left(\frac{\pi}{x}\right) = 3\frac{1}{7} \tag{22}$$

and to take n as the next largest natural number after the solution x. Making the same substitution, $y = \pi/x$ then equation (22) becomes

$$\tan y = k'y \text{ where } k' = \frac{3\frac{1}{7}}{\pi}.$$

This time the solution for y is approximately 0.0348164 and hence

$$x = \frac{\pi}{y} \cong 90.23 \text{ giving the smallest } n = 91.$$

As the values π/m and π/n are so small we could have estimated m and n by using

$$\sin x \cong x - \frac{x^3}{6} \quad \text{and} \quad \tan x \cong x + \frac{x^3}{3},$$

which are good approximations for small values of x. So solving (21) with the above approximation for sine, we find

$$x\left(\frac{\pi}{x} - \frac{\pi^3}{6x^3}\right) = 3\frac{10}{71}$$

giving

$$x \cong \sqrt{\frac{\pi^3}{6\left(\pi - 3\frac{10}{71}\right)}} \cong 83.14$$

Checking the truth of Archimedes inequality for $m = 82, 83, 84$ would reveal this to be a slight over-estimate.

4. Let $f(x) = \sec x$. Then, with some manipulation omitted, we find

$$\begin{aligned}
f(x) &= \sec x; \\
f'(x) &= \tan x \sec x; \\
f''(x) &= \sec^3 x + \sec x \tan^2 x; \\
f^{(3)}(x) &= 5\sec^3 x \tan x + \sec x \tan^3 x; \\
f^{(4)}(x) &= 5\sec^5 x + 18\sec^3 x \tan^2 x + \sec x \tan^4 x.
\end{aligned}$$

This means

$$f(0) = 1; \quad f'(0) = 0; \quad f''(0) = 1 \quad f^{(3)}(0) = 0; \quad f^{(4)}(0) = 5.$$

Hence the Taylor series for $\sec x$ begins

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$$

Alternatively, given that the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

is well-known, we could have combined this with the Binomial Theorem to obtain

$$\begin{aligned}
\sec x = &= \frac{1}{1 - \left(\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)} \\
&= 1 + \left(\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \left(\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)^2 + \cdots \\
&= 1 + \frac{x^2}{2} + \left(\frac{1}{24} + \frac{1}{2^2}\right)x^4 + \cdots \\
&= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots
\end{aligned}$$

Using this approximation we obtain

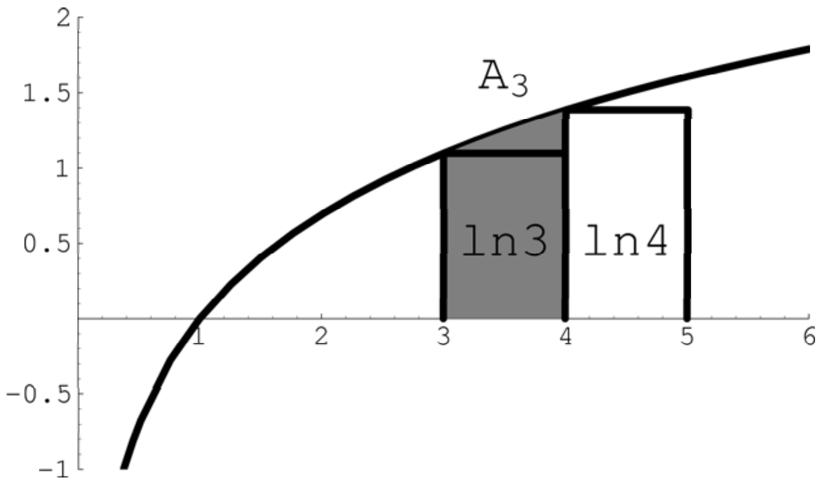
$$\sec(0.1) = 1 + 5 \times 0.01 + \frac{5}{24} \times 0.0001 = 1.0050208333\ldots$$

The value a calculator returns for $\sec(0.1)$ is $1.0050209184\ldots$, and so we see our estimate is accurate to 6 d.p.

5. Using integration by parts

$$\begin{aligned}\int \ln x \; dx &= \int 1 \times \ln x \; dx \\ &= x \times \ln x - \int x \times \frac{1}{x} \; dx \\ &= x \ln x - x + \text{const.}\end{aligned}$$

We now define $A_k = \int_k^{k+1} \ln x \; dx$ for each positive integer k. From the graph of $y = \ln x$ below, which we can see is increasing



we have

$$\ln k < A_k = \int_k^{k+1} \ln x \; dx < \ln (k+1) < \int_{k+1}^{k+2} \ln x \; dx.$$

Hence

$$\int_1^n \ln x \; dx < \ln 2 + \ln 3 + \cdots + \ln n < \int_2^{n+1} \ln x \; dx$$

and evaluating these integrals we have

$$n \ln n - n + 1 < \sum_1^n \ln r < (n+1) \ln (n+1) - (n+1) - 2 \ln 2 + 2 < (n+1) \ln (n+1) - n.$$

Now

$$\sum_1^n \ln r = \ln \left(\prod_1^n r \right) = \ln (n!) = \ln (G_n)^n = n \ln G_n.$$

So

$$\begin{aligned} & n \ln n - n + 1 < n \ln G_n < (n+1) \ln (n+1) - n \\ \Rightarrow \quad & \ln n - 1 + \frac{1}{n} < \ln G_n < \left(1 + \frac{1}{n}\right) \ln (n+1) - 1 \\ \Rightarrow \quad & \frac{1}{n} - 1 < \ln G_n - \ln n < \ln (n+1) + \frac{\ln (n+1)}{n} - \ln n - 1 \\ \Rightarrow \quad & \frac{1}{n} - 1 < \ln \left(\frac{G_n}{n}\right) < \ln \left(1 + \frac{1}{n}\right) + \frac{\ln (n+1)}{n} - 1. \end{aligned}$$

As n becomes very large then each of

$$\frac{1}{n}, \quad \ln \left(1 + \frac{1}{n}\right), \quad \frac{\ln (n+1)}{n}$$

becomes very close to 0, so that the LHS and RHS of the last inequality above both move toward -1. Therefore when n is very large we have from the aboe inequalities

$$\ln \left(\frac{G_n}{n}\right) \approx -1 \implies \frac{G_n}{n} \approx e^{-1} = \frac{1}{e}.$$

Solutions to CALCULUS EXERCISES 3 – Techniques of Integration

1. (i) As $\frac{1}{x}$ is the derivative of $\ln x$ we have

$$\int \frac{\ln x}{x}\, \mathrm{d}x = \int \ln x \, \mathrm{d}(\ln x) = \frac{1}{2}(\ln x)^2 + \text{const}.$$

(ii) By integration by parts, and recalling that the derivative of $\tan x$ is $\sec^2 x$, then

$$\begin{aligned}
\int x \sec^2 x \, \mathrm{d}x &= x \tan x - \int \tan x \, \mathrm{d}x \\
&= x \tan x - \int \frac{\sin x}{\cos x}\, \mathrm{d}x \\
&= x \tan x - \int \frac{-\mathrm{d}(\cos x)}{\cos x} \\
&= x \tan x + \ln|\cos x| + \text{const}.
\end{aligned}$$

(iii) Using partial fractions, we have

$$\begin{aligned}
\int_3^\infty \frac{1}{(x-1)(x-2)}\, \mathrm{d}x &= \int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x-1}\right) \mathrm{d}x \\
&= [\ln|x-2| - \ln|x-1|]_3^\infty \\
&= \left[\ln\left|\frac{x-2}{x-1}\right|\right]_3^\infty \\
&= \ln 1 - \ln\frac{1}{2} = \ln 2.
\end{aligned}$$

(iv) Using integration by parts, thinking of the integrand as $1 \times \tan^{-1} x$, we have

$$\begin{aligned}
\int_0^1 \tan^{-1} x \, \mathrm{d}x &= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, \mathrm{d}x \\
&= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 \\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2.
\end{aligned}$$

(v) Using a substitution of $u = e^x$, and noting $\mathrm{d}u = e^x \, \mathrm{d}x = u \, \mathrm{d}x$, then

$$\begin{aligned}
\int_0^1 \frac{1}{e^x+1}\, \mathrm{d}x &= \int_1^e \frac{1}{u+1}\, \frac{\mathrm{d}u}{u} \\
&= \int_1^e \left(\frac{1}{u} - \frac{1}{u+1}\right) \mathrm{d}u \\
&= [\ln|u| - \ln|u+1|]_1^e \\
&= 1 - \ln(1+e) + \ln 2.
\end{aligned}$$

2. (i) Making the substitution $x = \tan\theta$ (recalling that $1 + \tan^2\theta = \sec^2\theta$, so that this substitution leaves a square in the denominator), we obtain

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2+1} &= \int \frac{\sec^2\theta\ \mathrm{d}\theta}{\tan^2\theta + 1} \\
&= \int \frac{\sec^2\theta\ \mathrm{d}\theta}{\sec^2\theta} \\
&= \theta + \text{const.} = \tan^{-1} x + \text{const.}
\end{aligned}$$

(ii) Similarly, as $\cosh^2\theta - 1 = \sinh^2\theta$, making the substitution $x = \cosh\theta$ gives

$$\begin{aligned}
\int_1^2 \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \int_0^{\cosh^{-1}2} \frac{\sinh\theta\ \mathrm{d}\theta}{\sqrt{\cosh^2\theta - 1}} \\
&= \int_0^{\cosh^{-1}2} \frac{\sinh\theta\ \mathrm{d}\theta}{\sinh\theta} \\
&= \cosh^{-1} 2.
\end{aligned}$$

Alternatively, to do this integral we could also have noted that $\sec^2\theta - 1 = \tan^2\theta$; making the substitution $x = \sec\theta$ then gives

$$\begin{aligned}
\int_1^2 \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \int_0^{\sec^{-1}2} \frac{\tan\theta\sec\theta\ \mathrm{d}\theta}{\sqrt{\sec^2\theta - 1}} \\
&= \int_0^{\sec^{-1}2} \frac{\tan\theta\sec\theta}{\tan\theta}\ \mathrm{d}\theta \\
&= \int_0^{\sec^{-1}2} \sec\theta\ \mathrm{d}\theta \\
&= \left[\ln|\sec\theta + \tan\theta|\right]_0^{\sec^{-1}2} \\
&= \ln\left(2 + \tan\sec^{-1}2\right) \\
&= \ln(2+\sqrt{3}).
\end{aligned}$$

[Check, with a calculator, that these answers are equal; can you prove they are equal?]

(iii) Making the substitution $x = 2\sin\theta$ (recalling that $1 - \sin^2\theta = \cos^2\theta$, so that this substitution leaves a square under the root in the denominator), we obtain

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{4-x^2}} &= \int \frac{2\cos\theta\ \mathrm{d}\theta}{\sqrt{4 - 4\sin^2\theta}} \\
&= \int \frac{2\cos\theta\ \mathrm{d}\theta}{2\cos\theta} \\
&= \theta + \text{const.} = \sin^{-1}\frac{x}{2} + \text{const.}
\end{aligned}$$

(iv) As in the second integral, we make the substitution $x = \cosh\theta$. This gives

$$\begin{aligned}
\int_2^\infty \frac{\mathrm{d}x}{(x^2-1)^{3/2}} &= \int_{\cosh^{-1}2}^\infty \frac{\sinh\theta\ \mathrm{d}\theta}{\left(\sinh^2\theta\right)^{3/2}} \\
&= \int_{\cosh^{-1}2}^\infty \frac{\mathrm{d}\theta}{\sinh^2\theta} \\
&= [-\coth\theta]_{\cosh^{-1}2}^\infty \\
&= -1 + \coth\cosh^{-1}2 \\
&= -1 + \frac{2}{\sqrt{3}},
\end{aligned}$$

the last equality coming from the identity $\coth^2 u = \cosh^2 u/(\cosh^2 u - 1)$.

3. The idea, with the following integrand, is to first complete the square in the denominator, and then to use a trigonometric substitution, in a similar way to the previous question, to leave a square in the denominator.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{3x^2+2x+1} &= \frac{1}{3}\int \frac{\mathrm{d}x}{x^2+\frac{2x}{3}+\frac{1}{3}} \\
&= \frac{1}{3}\int \frac{\mathrm{d}x}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}} \\
&= \frac{1}{3}\int \frac{\frac{\sqrt{2}}{3}\sec^2\theta\ \mathrm{d}\theta}{\frac{2}{9}\left(1+\tan^2\theta\right)} \quad \text{[substituting } x+\frac{1}{3}=\frac{\sqrt{2}}{3}\tan\theta\text{]} \\
&= \frac{1}{3}\int \frac{3}{\sqrt{2}}\ \mathrm{d}\theta = \frac{1}{\sqrt{2}}\theta + \text{const.} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + \text{const.}
\end{aligned}$$

Similarly

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+5}} &= \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2+4}} \\
&= \int \frac{2\cosh\theta\ \mathrm{d}\theta}{\sqrt{4\cosh^2\theta}} \quad \text{[substituting } x+1=2\sinh\theta\text{]} \\
&= \theta + \ \text{const.} \\
&= \sinh^{-1}\frac{x+1}{2} + \text{const.}.
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{4x^2+4x+5} &= \int_0^\infty \frac{\mathrm{d}x}{(2x+1)^2+4} \\
&= \int_{\tan^{-1}2}^{\pi/2} \frac{\sec^2\theta\ \mathrm{d}\theta}{4\sec^2\theta} \quad \text{[substituting } 2x+1=2\tan\theta\text{]} \\
&= = \frac{\pi}{8} - \frac{1}{4}\tan^{-1}2.
\end{aligned}$$

4. Let $t=\tan\frac{\theta}{2}$. The third identity is a simple expression of the tangent double-angle formula. Remembering the equivalent formulas for sine and cosine we see

$$\frac{2t}{1+t^2} = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta;$$

$$\frac{1-t^2}{1+t^2} = \frac{1-\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta.$$

As $\theta = 2\tan^{-1}t$, then $\mathrm{d}\theta = 2\mathrm{d}t/(1+t^2)$ also.

Finally, making the $t=\tan\frac{\theta}{2}$ substitution in the given integral we see

$$\begin{aligned}
\int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1+\sin\theta)^2} &= \int_0^1 \frac{\frac{2\mathrm{d}t}{1+t^2}}{\left(1+\frac{2t}{1+t^2}\right)^2} \\
&= \int_0^1 \frac{2(1+t^2)}{(t^2+2t+1)^2}\ \mathrm{d}t \\
&= 2\int_0^1 \frac{1+t^2}{(t+1)^4}\ \mathrm{d}t \\
&= 2\int_0^1 \frac{(t+1)^2-2(t+1)+2}{(t+1)^4}\ \mathrm{d}t \\
&= 2\int_0^1 \left(\frac{1}{(t+1)^2} - \frac{2}{(t+1)^3} + \frac{2}{(t+1)^4}\right)\ \mathrm{d}t \\
&= 2\left[\frac{-1}{t+1} + \frac{1}{(t+1)^2} - \frac{2}{3(1+t)^3}\right]_0^1 \\
&= 2\left(-\frac{1}{2}+\frac{1}{4}-\frac{2}{24}+1-1+\frac{2}{3}\right) = \frac{2}{3}.
\end{aligned}$$

5. For n a natural number, let

$$I_n = \int_0^{\pi/2} x^n \sin x \ \mathrm{d}x.$$

Then

$$I_0 = \int_0^{\pi/2} \sin x \ \mathrm{d}x = [-\cos x]_0^{\pi/2} = 1;$$

$$I_1 = \int_0^{\pi/2} x \sin x \ \mathrm{d}x = [-x\cos x + \sin x]_0^{\pi/2} = 1.$$

If we use integration by parts to repeatedly integrate to trigonometric part of the integrand, whilst differentiating down the powers of x, we can hope to determine a *reduction formula* for I_n. So

$$\begin{aligned} I_n &= \int_0^{\pi/2} x^n \sin x \ \mathrm{d}x \\ &= [-x^n \cos x]_0^{\pi/2} + \int_0^{\pi/2} n x^{n-1} \cos x \ \mathrm{d}x \\ &= 0 + n \int_0^{\pi/2} x^{n-1} \cos x \ \mathrm{d}x \\ &= n \left\{ \left[x^{n-1} \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} (n-1) x^{n-2} \sin x \ \mathrm{d}x \right\} \\ &= n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}, \end{aligned}$$

as required. From this reduction formula we can determine

$$\begin{aligned} I_5 &= 5\left(\frac{\pi}{2}\right)^4 - 20 I_3 \\ &= \frac{5\pi^4}{16} - 20\left(\frac{3\pi^2}{4} - 6 I_1\right) \\ &= \frac{5\pi^4}{16} - 15\pi^2 + 120; \\ I_6 &= 6\left(\frac{\pi}{2}\right)^5 - 30 I_4 \\ &= \frac{3\pi^5}{16} - 30\left(\frac{\pi^3}{2} - 12 I_2\right) \\ &= \frac{3\pi^5}{16} - 15\pi^3 + 360(\pi - 2) \\ &= \frac{3\pi^5}{16} - 15\pi^3 + 360\pi - 720. \end{aligned}$$

Solutions to CALCULUS EXERCISES 4 – Differential Equations

1. The three given differential equations are all separable.

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} \implies \int y\ \mathrm{d}y = \int x^2\ \mathrm{d}x \implies \frac{1}{2}y^2 = \frac{1}{3}x^3 + c;$$

(ii)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 x}{\cos^2 2y} \quad &\implies \int \cos^2 2y\ \mathrm{d}y = \int \cos^2 x\ \mathrm{d}x \\ &\implies \tfrac{1}{2}\int (\cos 4y + 1)\ \mathrm{d}y = \tfrac{1}{2}\int (\cos 2x + 1)\ \mathrm{d}x \\ &\implies \frac{\sin 4y}{4} + y = \frac{\sin 2x}{2} + x + c; \end{aligned}$$

(iii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+2y} \implies \int e^{-2y}\ \mathrm{d}y = \int e^x\ \mathrm{d}x \implies \frac{-1}{2}e^{-2y} = e^x + c.$$

2. Initial value problems – again the equations are separable.

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-2x}{y} \implies \int y\ \mathrm{d}y = \int (1-2x)\ \mathrm{d}x \implies \tfrac{1}{2}y^2 = x - x^2 + c.$$

But $y(1) = -2$ and so $c = 2$. Hence

$$x^2 - x + \tfrac{1}{2}y^2 = 2 \implies \left(x - \tfrac{1}{2}\right)^2 + \tfrac{1}{2}y^2 = \frac{9}{4}.$$

Then

$$y = -\sqrt{\frac{9}{2} - 2\left(x - \tfrac{1}{2}\right)^2};$$

which is part of an ellipse. The solution is valid for $-1 < x < 2$.

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(x^2+1)}{4y^3} \implies \int 4y^3\ \mathrm{d}y = \int (x^3 + x)\ \mathrm{d}x \implies y^4 = \frac{x^4}{4} + \frac{x^2}{2} + c.$$

As $y(0) = \frac{-1}{\sqrt{2}}$ then $c = \frac{1}{4}$. So

$$y^4 = \frac{1}{4}(x^2+1)^2 \implies y = -\sqrt{\frac{x^2+1}{2}}.$$

This is part of a hyperbola. The solution is valid for all x.

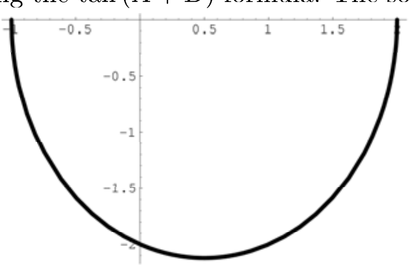
(iii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y^2}{1+x^2} \implies \int \frac{\mathrm{d}y}{1+y^2} = \int \frac{\mathrm{d}x}{1+x^2} \implies \tan^{-1} y = \tan^{-1} x + c$$

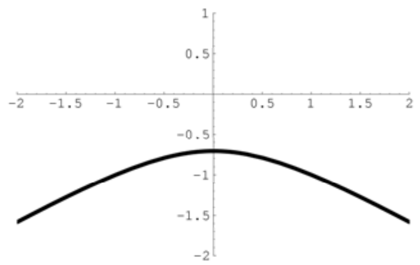
As $y(0) = 1$ then $c = \frac{\pi}{4}$. So

$$y = .\tan\left(\tan^{-1} x + \frac{\pi}{4}\right) = \frac{x+1}{1-x}$$

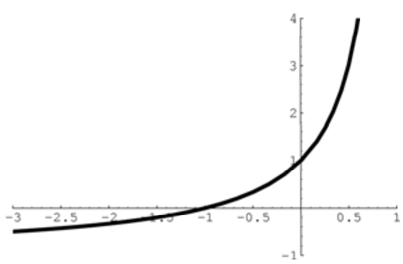
using the $\tan(A+B)$ formula. The solution is valid for $x < 1$.



$$y = -\sqrt{\frac{9}{2} - 2\left(x - \tfrac{1}{2}\right)^2};$$



$$y = -\sqrt{\frac{x^2+1}{2}}.$$



$$y = (1+x)(1-x)^{-1}.$$

3. The function $x(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x,$$

where ω is a constant. Let $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ so that

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

by the chain rule. The equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\omega^2 x$$

is separable:

$$\int v \ \mathrm{d}v = -\omega^2 \int x \ \mathrm{d}x \implies v^2 = -\omega^2 x^2 + c,$$

for some constant c. As $v = 0$ when $x = a$, then $c = a^2\omega^2$.

Now the differential equation

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \omega\sqrt{a^2 - x^2}$$

yields

$$\omega t + \epsilon = \int \omega \ \mathrm{d}t = \int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a},$$

for some constant ϵ. [Those who have not met the 'standard' integral above can use the substitution $x = a\sin\theta$ to evaluate it.] Solving for x gives

$$x = a\sin(\omega t + \epsilon).$$

4. (i) The auxiliary equation of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y = 0$$

is $m^2 = 1$. This has roots $m = 1$ and $m = -1$, and so the general solution is

$$y = Ae^x + Be^{-x}.$$

(ii) The auxiliary equation of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 0$$

is $m^2 + 4 = 0$ which has roots $m = 2i$ and $m = -2i$. The equation's general solution is then

$$y = A\cos 2x + B\sin 2x.$$

The initial conditions are $y(0) = A = 1$ and $y'(0) = 2B = 1$, so that

$$y = \cos 2x + \tfrac{1}{2}\sin 2x.$$

(iii) The auxiliary equation of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

is $m^2 + 3m + 2 = 0$ which has roots $m = -1$ and $m = -2$. So the general solution is

$$y = Ae^{-x} + Be^{-2x}.$$

(iv) The auxiliary equation of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$$

is $m^2 - 4m + 4 = 0$ which has a repeated root of $m = 2$. So the general solution is

$$y = (Ax + B)e^{2x}.$$

The initial conditions are $y(0) = B = 1$ and $y'(0) = A + 2B = 1$, so that $A = -1$. Hence

$$y = (1 - x)e^{2x}.$$

5. We are given that

$$(2x + y) + (x + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}F(x, y)}{\mathrm{d}x}$$

where $F(x, y)$ is a polynomial in x and y. Note that $x^a y^b$ differentiates to

$$ax^{a-1}y^b + bx^a y^{b-1}\frac{\mathrm{d}y}{\mathrm{d}x};$$

these terms have degree $a + b - 1$ in x and y together, one less than that of $x^a y^b$. Consequently, the x and y termsin the differential equation must have come from terms such as x^2, xy, y^2. Now

$$\frac{\mathrm{d}(ax^2 + bxy + cy^2)}{\mathrm{d}x} = (2ax + by) + (bx + 2cy)\frac{\mathrm{d}y}{\mathrm{d}x}$$

from which we see that $a = b = c = 1$. Hence the general solution of the equation is

$$x^2 + xy + y^2 = c.$$

After some inspection of

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)$$

we can see that

$$\frac{\mathrm{d}(x^2 e^y)}{\mathrm{d}x} = 2xe^y + x^2 e^y\frac{\mathrm{d}y}{\mathrm{d}x} \text{ and } \frac{\mathrm{d}(y\sin x)}{\mathrm{d}x} = y\cos x + \sin x\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The remaining $-\mathrm{d}y/\mathrm{d}x$ term is simply the derivative of y. So the differential equation's general solution is

$$x^2 e^y + y\sin x - y = c.$$

Equations that can be put into the form

$$\frac{\mathrm{d}F(x, y)}{\mathrm{d}x} = 0$$

are called *exact* differential equations.

Solutions to CALCULUS EXERCISES 5 – Further Differential Equations

1. The three differential equations are all separable.
(i)

$$\frac{dy}{dx} = \frac{y - xy}{xy - x} = \frac{y(x-1)}{x(y-1)} \implies \int \frac{y-1}{y}\,dy = \int \frac{x-1}{x}\,dx.$$

So the general solution is

$$y - \ln y = x - \ln x + c.$$

(ii)

$$\frac{dy}{dx} = \frac{\sin^{-1} x}{y^2\sqrt{1-x^2}} = \frac{\sin^{-1} x}{y^2}\frac{d(\sin^{-1} x)}{dx}.$$

Hence

$$\frac{1}{3}y^3 = \int y^2\,dy = \int \sin^{-1} x\,d(\sin^{-1} x) = \tfrac{1}{2}(\sin^{-1} x)^2 + c.$$

As $y(0) = 0$ then $c = 0$, and so we see the solution is

$$y^3 = \frac{3}{2}(\sin^{-1} x)^2 \text{ for } -1 < x < 1.$$

(iii) The given differential equation is separable in dy/dx and x. So setting $z = dy/dx$ we see

$$\frac{dz}{dx} = (1+3x^2)z^2 \implies \int \frac{dz}{z^2} = \int (1+3x^2)\,dx \implies \frac{-1}{z} = x + x^3 + c.$$

As $z = -1/2$ when $x = 1$ then $c = 0$. Recalling $z = dy/dx$ we have

$$\frac{dy}{dx} = \frac{-1}{x+x^3} = \frac{x}{1+x^2} - \frac{1}{x} \implies y = \tfrac{1}{2}\ln(1+x^2) - \ln x + c.$$

As $y = 0$ when $x = 1$ then $c = -\frac{1}{2}\ln 2$. So

$$y = \tfrac{1}{2}\ln(1+x^2) - \ln x - \tfrac{1}{2}\ln 2 = \tfrac{1}{2}\ln\left(\frac{1+x^2}{2x^2}\right). \text{ for } x > 0.$$

2. (i) The integrating factor of

$$\frac{dy}{dx} + xy = x$$

is $\exp\int x\,dx = \exp\left(\frac{x^2}{2}\right)$, and so multiplying through and integrating:

$$\frac{d}{dx}\left(e^{x^2/2}y\right) = e^{x^2/2}\frac{dy}{dx} + xe^{x^2/2}y = xe^{x^2/2} \implies e^{x^2/2}y = e^{x^2/2} + c.$$

As $y(0) = 0$ then $c = -1$. Hence

$$y = 1 - e^{-x^2/2}.$$

(ii) The integrating factor of

$$\frac{dy}{dx} + \frac{-3}{2x}y = \frac{1}{2x^3}$$

is $\exp\int(-3/2x)\,dx = \exp(-3\ln x/2) = x^{-3/2}$. Multiplying through by this, and integrating, we find

$$\frac{d(x^{-3/2}y)}{dx} = x^{-3/2}\frac{dy}{dx} - \frac{3y}{2x^{5/2}} = \frac{1}{2x^{9/2}} \implies x^{-3/2}y = \frac{-1}{7}x^{-7/2} + c.$$

As $y = 0$ when $x = 1$ then $c = 1/7$. Hence

$$y = \frac{x^{3/2} - x^{-2}}{7} \text{ for } x > 0.$$

(iii) The integrating factor in

$$\frac{dy}{dx} - y\tan x = 1$$

is $\exp\int(-\tan x)\,dx = \exp\ln\cos x = \cos x$. Multiplying through by this, and integrating, shows

$$\frac{d}{dx}(y\cos x) = \cos x\frac{dy}{dx} - y\sin x = \cos x \implies y\cos x = \sin x + c.$$

As $y = 1$ when $x = 0$ then $c = 1$. Hence

$$y = \tan x + \sec x.$$

3. The auxiliary equation of the homogeneous differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

is $m^2 + 3m + 2 = 0$ which has roots $m = -1$ and $m = -2$. So its general solution is

$$y = Ae^{-x} + Be^{-2x}.$$

It remains, for the four given *inhomogeneous* equations, to find a particular solution to each.

(i) Try $y = ax + b$. Then, comparing coefficients of x^1, x^0 in

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0 + 3a + 2(ax + b) = x,$$

we see $2a = 1$ and $3a + 2b = 0$. So $a = 1/2$ and $b = -3/4$. The equation's general solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{x}{2} - \frac{3}{4}.$$

(ii) Try $y = a \sin x + b \cos x$. Then, comparing coefficients of $\cos x, \sin x$ in

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = (-a \sin x - b \cos x) + 3(a \cos x - b \sin x) + 2(a \sin x + b \cos x) = \sin x,$$

we see $a - 3b = 1$ and $b + 3a = 0$. So $a = 1/10$ and $b = -3/10$ and the equation's general solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{10}\sin x - \frac{3}{10}\cos x.$$

(iii) Try $y = ae^x$. Then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = ae^x + 3ae^x + 2ae^x = 6ae^x = e^x,$$

and hence $a = 1/6$. The equation's general solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{6}e^x.$$

(iv) As e^{-x} is already a solution of the homogeneous differential equation then there is no point trying $y = ae^{-x}$, as it would just yield zero. Instead we try $y = axe^{-x}$. Then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = a(x-2)e^{-x} + 3a(1-x)e^{-x} + 2axe^{-x} = ae^{-x} = e^{-x}$$

and hence $a = 1$. The equation's general solution is

$$y = Ae^{-x} + Be^{-2x} + xe^{-x}.$$

4. (a) Substituting $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ in the differential equation gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{xy} \implies v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x^2 + v^2x^2}{vx^2} = \frac{1 + v^2}{v} \implies x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{v}$$

which is separable. So

$$\tfrac{1}{2}v^2 = \int v \, \mathrm{d}v = \int \frac{\mathrm{d}x}{x} = \ln x + c.$$

Substituting $v = y/x$ back in gives

$$y^2 = 2x^2(\ln x + c).$$

Similarly in the second equation we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \implies v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sqrt{1 + v^2} \implies x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{1 + v^2}$$

which is separable. Then

$$\sinh^{-1} x = \int \frac{\mathrm{d}v}{\sqrt{1 + v^2}} = \int \frac{\mathrm{d}x}{x} = \ln x + c.$$

As $\sinh^{-1} v = \ln(v + \sqrt{v^2+1})$ our solution can be rewritten as

$$v + \sqrt{v^2+1} = Ax \implies y + \sqrt{x^2+y^2} = Ax^2.$$

(b) If we substitute $x = X + a, y = Y + b$ into the differential equation we see

$$\frac{dy}{dx} = \frac{x = y - 3}{x - y - 1} \implies \frac{dY}{dX} = \frac{X + Y + (a + b - 3)}{X - Y + (a - b - 1)}$$

which is homogeneous polar when $a + b = 3$ and $a - b = 1$ – i.e when $a = 2$ and $b = 1$.

Substituting $Y = VX$ into this differential equation

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \implies V + X\frac{dV}{dX} = \frac{X + VX}{X - VX} = \frac{1+V}{1-V} \implies X\frac{dV}{dX} = \frac{1+V^2}{1-V}$$

which is separable. So

$$\ln X = \int \frac{dX}{X} = \int \frac{1-V}{1+V^2}\, dV = \tan^{-1} V - \frac{1}{2}\ln\left(1+V^2\right) + c.$$

and substituting $V = Y/X$ back in gives

$$\frac{1}{2}\ln\left(X^2+Y^2\right) = \tan^{-1}\frac{Y}{X} + c.$$

Recalling $X = x - 2$ and $Y = y - 1$ we have the general solution

$$\frac{1}{2}\ln\left(x^2 - 4x + y^2 - 2y + 5\right) = \tan^{-1}\frac{y-2}{x-1} + c.$$

5. A particle moving in the xy-plane, at time t is at $(x(t), y(t)$ where

$$\frac{dy}{dt} = x + y, \qquad \frac{dx}{dt} = x - y \implies \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{x+y}{x-y}.$$

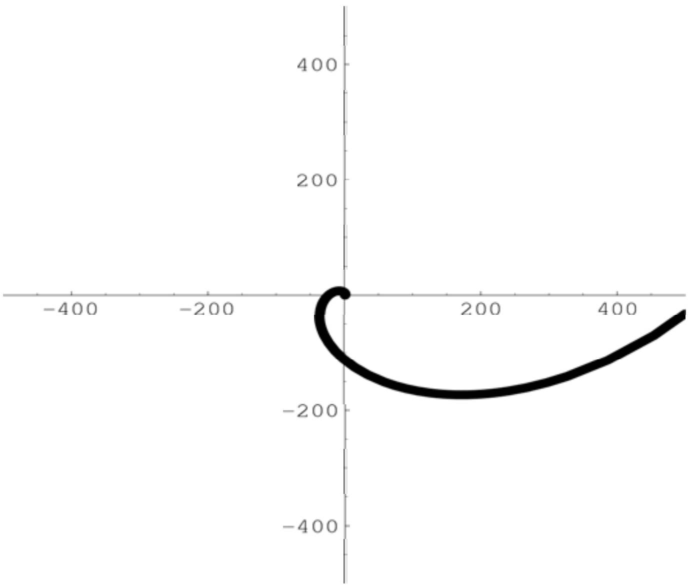
From the previous question we know that the general solution to this homogeneous polar equation is

$$\tfrac{1}{2}\ln(x^2+y^2) = \tan^{-1}\frac{y}{x} + c.$$

As $y = 0$ when $x = 1$ then $c = 0$. Now changing to polar co-ordinates gives

$$\ln r = \tfrac{1}{2}\ln r^2 = \theta \implies r = e^\theta.$$

This is what is called the *logarithmic spiral*. Putting $x = e^\theta \cos\theta$ and $y = e^\theta \sin\theta$ into either of the defining differential equations we see that $\theta = t$. So the particle moves around the spiral at ever increasing speeds, but always taking the same time, 2π, to loop the origin.



The particle's path for $0 \leqslant t \leqslant 2\pi$

Solutions to COMPLEX NUMBERS EXERCISES

1. Write $\omega = a + ib$. If $\omega^2 = -5 - 12i$ then

$$\omega^2 = (a+ib)^2 = a^2 + 2abi + i^2b^2 = (a^-b^2) + 2abi = -5 - 12i.$$

Comparing real and imaginary parts we see that

$$a^2 - b^2 = -5 \qquad \text{and} \qquad 2ab = -12.$$

So

$$a^2 - (-6/a)^2 = -5 \implies a^4 + 5a^2 - 36 = 0 \implies (a^2-4)(a^2+9) = 0.$$

As $a^2 = -9$ has no real solutions we see $a = \pm 2$ and $b = \mp 3$. So the two square roots of $-5 - 12i$ are $\pm(2 - 3i)$.

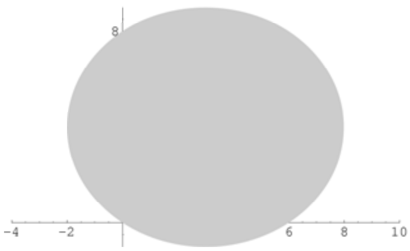
Applying the quadratic formula to

$$z^2 - (4+i)z + (5+5i) = 0$$

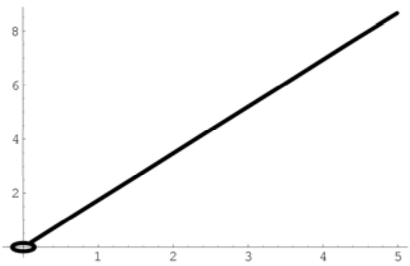
yields

$$\begin{aligned} z &= \frac{(4+i) \pm \sqrt{(4+i)^2 - 4(5+5i)}}{2} \\ &= \frac{(4+i) \pm \sqrt{-5-12i}}{2} = \frac{(4+i) \pm (2-3i)}{2} = 3 - i \text{ or } 1 + 2i. \end{aligned}$$

2.



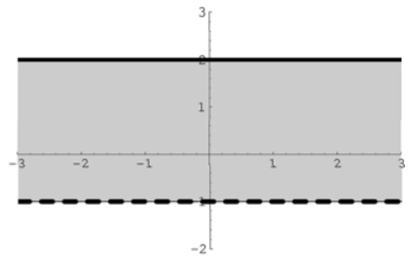
Recall that $|z - a|$ is the distance of the point (represented by the complex number) z from the point a. So this inequality is satisfied by those points which are less than 5 units distance from the point $3 + 4i$.



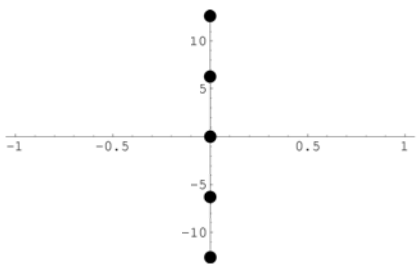
The argument of a complex number z is the angle $0z$ makes with the positive real axis. So this is the half-line

$$y = x \tan \frac{\pi}{3} = \sqrt{3}x/2 \text{ for } x > 0.$$

Note that this doesn't include the origin, whose argument is undefined.



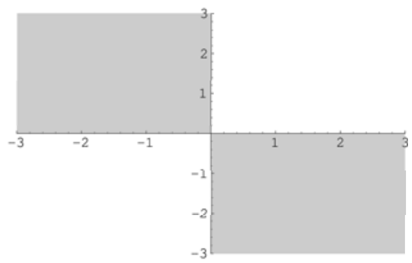
Now putting $z = x+iy$ into $((iz+3)/2)$ gives $((3-y+ix)/2)$ which has real part $(3-y)/2$. So this is the region $0 \leqslant (3-y)/2 < 2$ or equivalently $-1 < y \leqslant 3$.



Given a complex number $z = x + iy$ then

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y)$$

has modulus e^x and argument y.
As 1 has modulus 1 and argument 0 (up to multiples of 2π), then $x = 0$ and $y = 2n\pi$ for some integer n.
So the solutions of $e^z = 1$ are $z = 2n\pi i$ for $n \in \mathbb{Z}$.



Putting $z = x + iy$ again we have

$$\text{Im}(z^2) = \text{Im}((x+iy)^2) = \text{Im}((x^2 - y^2) + 2xyi) = 2xy < 0$$

means that x and y must have differing signs, i.e. the two quadrants given by $x > 0, y < 0$ or $x < 0, y > 0$.
Alternatively, we could have set $z = re^{i\theta}$, which leads to

$$\text{Im}(z^2) = \text{Im}(r^2 e^{2i\theta}) = r^2 \sin 2\theta$$

which is negative when $\sin 2\theta = 0$, giving the same two quadrants again.

3. Let $z_0 = x(t) + iy(t) = 2 + it$. Then

- $iz_0 = -t + 2i$ which as t varies maps out the line $y = 2$;
- $(z_0)^2 = (4 - t^2) + 4it$ and eliminating t gives $x = 4 - \left(\frac{y}{4}\right)^2$;
- $e^{z_0} = e^2(\cos t + i\sin t)$ which as t varies maps out the circle $x^2 + y^2 = e^4$;
- $\frac{1}{z_0} = \frac{2-it}{4+t^2}$ and eliminating $t = -2y/x$ gives $x = \frac{2}{4+4y^2/x^2} = \frac{x^2}{2(x^2+y^2)}$ and rearranging gives

$$x^2 + y^2 = \frac{x}{2} \implies \left(x - \frac{1}{4}\right)^2 + y^2 = \left(\frac{1}{4}\right)^2.$$

As t varies then $2 + it$ maps out the line $\text{Re}z = 2$, and similarly varying t in the above image points maps out all of the image of this line. Note that in the final part the origin is omitted from the circle, as this corresponds to a point $t = \infty$ which is not on the original line.

4. (a) As $e^{i\theta} = \cos\theta + i\sin\theta$ then $\cos\theta = \text{Re}(e^{i\theta})$. So

$$\begin{aligned}\cos(\alpha+\beta) &= \text{Re}(e^{i(\alpha+\beta)}) \\ &= \text{Re}(e^{i\alpha}e^{i\beta}) \\ &= \text{Re}[(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)] = \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

(b) *De Moivre's Theorem* states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Hence, by the Binomial Theorem,

$$\begin{aligned}\cos 5\theta &= \text{Re}(\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

5. (a) Let $z = \cos\theta + i\sin\theta$. Then

$$\frac{1}{z} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta.$$

Solving for $\cos\theta$ and $\sin\theta$ from these expressions for z and $1/z$ gives

$$2\cos\theta = z + \frac{1}{z} \qquad \text{and} \qquad 2i\sin\theta = z - \frac{1}{z}.$$

By De Moivre's Theorem $z^n = \cos n\theta + i\sin n\theta$. Hence

$$2\cos n\theta = z^n + \frac{1}{z^n} \qquad \text{and} \qquad 2i\sin n\theta = z^n - \frac{1}{z^n}.$$

(b) Hence

$$\begin{aligned}32\cos^5\theta &= (z + z^{-1})^5 \\ &= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} \\ &= = (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}) \\ &= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta,\end{aligned}$$

to give the required result. Finally,

$$\begin{aligned}\int_0^{\pi/2}\cos^5\theta\ \mathrm{d}\theta &= \frac{1}{16}\int_0^{\pi/2}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\ \mathrm{d}\theta \\ &= \frac{1}{16}\left[\frac{\sin 5\theta}{5} + \frac{5\sin 3\theta}{3} + 10\sin\theta\right]_0^{\pi/2} \\ &= \frac{1}{16}\left(\frac{1}{5} - \frac{5}{3} + 10\right) \\ &= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}.\end{aligned}$$

Solutions to GEOMETRY EXERCISES

1.

- $$\begin{aligned} \mathbf{r} \wedge (\mathbf{i}+\mathbf{j}) = (\mathbf{i}-\mathbf{j}) &\iff (x,y,z)\wedge(1,1,0) = (1,-1,0) \\ &\iff (-z,z,x-y) = (1,-1,0) \\ &\iff z=-1 \text{ and } x=y \text{ [this is the equation of a line]} \end{aligned}$$

- $$\begin{aligned} \mathbf{r}\cdot\mathbf{i} = 1 &\iff (x,y,z)\cdot(1,0,0) = 1 \\ &\iff x = 1 \text{ [this is the equation of a plane]} \end{aligned}$$

- $$\begin{aligned} |\mathbf{r}-\mathbf{i}| = |\mathbf{r}-\mathbf{j}| &\iff |\mathbf{r}-\mathbf{i}|^2 = |\mathbf{r}-\mathbf{j}|^2 \\ &\iff |(x-1,y,z)|^2 = |(x,y-1,z)|^2 \\ &\iff (x-1)^2+y^2+z^2 = x^2+(y-1)^2+z^2 \\ &\iff 1-2x = 1-2y \\ &\iff x = y \text{ [a plane; the bisector of the line segment } \mathbf{i} \text{ to } \mathbf{j}] \end{aligned}$$

- $$\begin{aligned} |\mathbf{r}-\mathbf{i}| = 1 &\iff |\mathbf{r}-\mathbf{i}|^2 = 1 \\ &\iff |(x-1,y,z)|^2 = 1 \\ &\iff (x-1)^2+y^2+z^2 = 1 \text{ [a sphere with unit radius and centre } (1,0,0)] \end{aligned}$$

- $$\begin{aligned} \mathbf{r}\cdot\mathbf{i} = \mathbf{r}\cdot\mathbf{j} = \mathbf{r}\cdot\mathbf{k} \quad &iff \quad (x,y,z)\cdot(1,0,0) = (x,y,z)\cdot(0,1,0) = (x,y,z)\cdot(0,0,1) \\ &\iff x = y = z \text{ [the equation of a line]} \end{aligned}$$

- $$\begin{aligned} \mathbf{r}\wedge\mathbf{i} = \mathbf{i} &\iff (x,y,z)\wedge(1,0,0) = (1,0,0) \\ &\iff (0,z,-y) = (1,0,0) \text{ [true for no } x,y,z\text{; the empty set]} \end{aligned}$$

2. We can parametrise the lines $(x-1)/2 = (y-3)/3 = z/2$ and $x = 2, (y-1)/2 = z$ as

$$\mathbf{r}(\lambda) = (1,3,0) + \lambda(2,3,2) \quad \text{and} \quad \mathbf{s}(\mu) = (2,1,0) + \mu(0,2,1).$$

These lines are in the directions $(2,3,2)$ and $(0,2,1)$ respectively. The vector connecting $\mathbf{r}(\lambda)$ and $\mathbf{s}(\mu)$ equals

$$\mathbf{r}(\lambda) - \mathbf{s}(\mu) = (-1+2\lambda, 2+3\lambda-2\mu, 2\lambda-\mu).$$

This vector is perpendicular to the two lines when

$$(-1+2\lambda, 2+3\lambda-2\mu, 2\lambda-\mu)\cdot(2,3,2) = -2+4\lambda+6+9\lambda-6\mu+4\lambda-2\mu = 4+17\lambda-8\mu = 0; \tag{23}$$
$$(-1+2\lambda, 2+3\lambda-2\mu, 2\lambda-\mu)\cdot(0,2,1) = 4+6\lambda-4\mu+2\lambda-\mu = 4+8\lambda-5\mu = 0. \tag{24}$$

Solving these simultaneous equations, we find $\lambda = 4/7$ and $\mu = 12/7$, which mean that the two points on the lines, which are closest, are $\mathbf{r}(4/7)$ and $\mathbf{s}(12/7)$. The vector between these two points is

$$\mathbf{r}\left(\frac{4}{7}\right) - \mathbf{s}\left(\frac{12}{7}\right) = \left(-1+\frac{8}{7}, 2+\frac{12}{7}-\frac{24}{7}, \frac{8}{7}-\frac{12}{7}\right) = \left(\frac{1}{7}, \frac{2}{7}, \frac{-4}{7}\right),$$

which has length

$$\frac{1}{7}\sqrt{1^2+2^2+4^2} = \frac{1}{7}\sqrt{21} = \sqrt{\frac{3}{7}}.$$

3. Let L_θ denote the line through (a,b) making an angle θ with the x-axis. This line can be parametrised as

$$\mathbf{r}(\lambda) = (a,b) + \lambda(\cos\theta, \sin\theta) = (a + \lambda\cos\theta, b + \lambda\sin\theta).$$

This meets the curve $y = x^2$ when

$$b + \lambda\sin\theta = (a + \lambda\cos\theta)^2 \Rightarrow \lambda^2(\cos^2\theta) + \lambda(2a\cos\theta - \sin\theta) + (a^2 - b) = 0.$$

If L_θ is tangential to $y = x^2$, then the quadratic in λ above will have a repeated root, and so a zero discriminant "$b^2 - 4ac$". So

$$\begin{aligned} \text{“}b^2 - 4ac\text{”} &= (2a\cos\theta - \sin\theta)^2 - 4(a^2 - b)\cos^2\theta \\ &= 4a^2\cos^2\theta - 4a\sin\theta\cos\theta + \sin^2\theta - 4a^2\cos^2\theta + 4b\cos^2\theta \\ &= \sin^2\theta - 4a\sin\theta\cos\theta + 4b\cos^2\theta = 0; \\ \Longrightarrow &\quad \tan^2\theta - 4a\tan\theta + 4b = 0. \end{aligned} \tag{25}$$

Given from a fixed point (a,b) there will be two lines which are tangential to the parabola. These will make angle θ_1 and θ_2 where $\tan\theta_1$ and $\tan\theta_2$ are the roots of the equation (25). Hence

$$\tan\theta_1 + \tan\theta_2 = 4a \quad \text{and} \quad \tan\theta_1\tan\theta_2 = 4b$$

because the sum and product of a quadratic's roots can be got from the formula

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta.$$

If these tangents make an angle $\theta_1 - \theta_2 = \pm\pi/4$, then

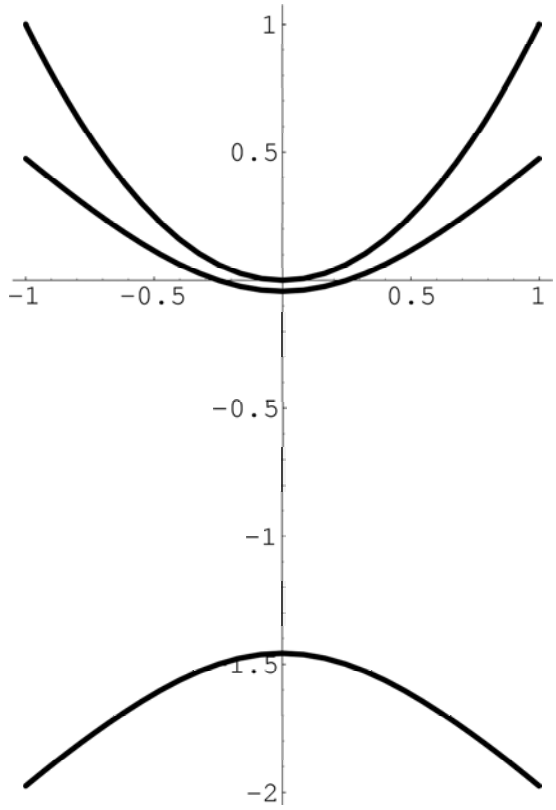
$$\pm 1 = \tan\left(\pm\frac{\pi}{4}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2}.$$

Hence

$$(1 + \tan\theta_1\tan\theta_2)^2 = (\tan\theta_1 - \tan\theta_2)^2 = (\tan\theta_1 + \tan\theta_2)^2 - 4\tan\theta_1\tan\theta_2;$$

so

$$\begin{aligned} & (1+4b)^2 = (4a)^2 - 4(4b) \\ \Longrightarrow\quad & 1 + 8b + 16b^2 = 16a^2 - 16b \\ \Longrightarrow\quad & 1 + 24b + 16b^2 = 16a^2. \end{aligned}$$



Graphs of the parabola $y = x^2$ and the hyperbola $1 + 24y + 16y^2 = 16x^2$.

4.

- The first map is $(x,y)^T \mapsto (x,-y)^T$, which is clearly reflection in the y-axis. Performing this reflection twice leaves everything fixed – so this map is invertible, in fact being its own inverse. We could also have show it is invertible by noting its determinant of -1 is non-zero.

- The second map is $(x,y)^T \mapsto (2x,y)^T$, is a stretch parallel to the x-axis with scale factor 2. Its inverse is $(x,y)^T \mapsto (x/2,y)^T$, a similar stretch now with scale factor $1/2$. Again we could also have show it is invertible by noting its determinant of 2 is non-zero.

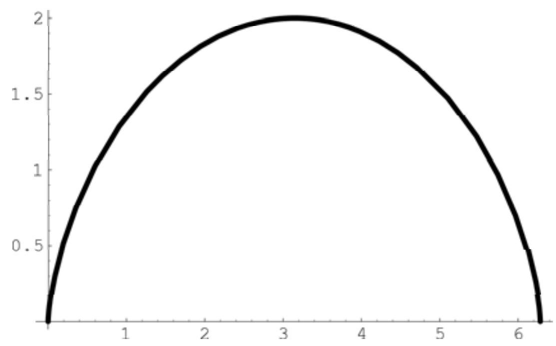
- The third map is $(x,y)^T \mapsto ((x+y)/2,(x+y)/2)$. Note that the image point lies on the line $y=x$ and so this map is not invertible. Geometrically the point $((x+y)/2,(x+y)/2)$ is the orthogonal (i.e. perpendicular) projection of the point (x,y) onto the line $y=x$. We could of course have shown it is not invertible by noting its matrix has zero determinant.

- The effect of the fourth map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  is to rotate the xy-plane around the origin anti-clockwise through an angle of θ. One way to appreciate this, is to note that the map takes $(r\cos\phi, r\sin\phi)$ to $(r\cos(\theta+\phi), r\sin(\theta+\phi))$. [Check this!] Rotating through θ about the origin in a clock-wise direction would invert this map. Alternatively we could note its determinant is 1.

5. (a)



The cycloid

(b) Arc-length $s(t)$ satisfies the differential equation

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (1-\cos t)^2 + \sin^2 t = 2 - 2\cos t = 4\sin^2 \tfrac{1}{2}t.$$

Hence the cycloid's arc-length equals

$$\int_0^{2\pi} \left|2\sin\tfrac{1}{2}t\right| \,\mathrm{d}t = \left[-4\cos\tfrac{1}{2}t\right]_0^{2\pi} = 8.$$

(c) The area bounded between the cycloid and the x-axis equals

$$\begin{aligned}
\int_{t=0}^{t=2\pi} y\,\mathrm{d}x &= \int_0^{2\pi} y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (1-2\cos t+\cos^2 t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \tfrac{1}{2}\cos 2t\right)\mathrm{d}t \\
&= \left[\frac{3t}{2} - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi.
\end{aligned}$$

(d) The area of the surface of revolution generated by the cycloid equals

$$
\begin{aligned}
2\pi \int_{t=0}^{t=2\pi} y \; \mathrm{d}s &= 2\pi \int_0^{2\pi} y \frac{\mathrm{d}s}{\mathrm{d}t} \; \mathrm{d}t \\
&= 2\pi \int_0^{2\pi} (1-\cos t) \times 2\sin \tfrac{1}{2}t \; \mathrm{d}t \\
&= 2\pi \int_0^{2\pi} 4(1-\cos^2 \tfrac{1}{2}t) \sin \tfrac{1}{2}t \; \mathrm{d}t \\
&= 2\pi \left[-8\cos \tfrac{1}{2}t + \frac{8}{3} \cos^3 \tfrac{1}{2}t \right]_0^{2\pi} \\
&= = 2\pi \left(8 - \frac{8}{3} + 8 - \frac{8}{3} \right) = \frac{64\pi}{3}.
\end{aligned}
$$

(e) The volume enclosed by the surface of revolution generated by the cycloid equals

$$
\begin{aligned}
\pi \int_{t=0}^{t=2\pi} y^2 \; \mathrm{d}x &= \pi \int_0^{2\pi} y^2 \frac{\mathrm{d}x}{\mathrm{d}t} \; \mathrm{d}t \\
&= \pi \int_0^{2\pi} (1-\cos t)^2 (1-\cos t) \; \mathrm{d}t \\
&= \pi \int_0^{2\pi} (1-\cos t)^3 \; \mathrm{d}t \\
&= \pi \int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t) \; \mathrm{d}t \\
&= \pi \int_0^{2\pi} \left(1 - 3\cos t + \frac{3}{2}[\cos 2t + 1] - \frac{1}{4}[\cos 3t + 3\cos t] \right) \; \mathrm{d}t \\
&= \pi \int_0^{2\pi} \left(\frac{5}{2} - \frac{15}{4}\cos t + \frac{3}{2}\cos 2t - \frac{1}{4}\cos 3t \right) \; \mathrm{d}t \\
&= \pi \left[\frac{5t}{2} - \frac{15}{4}\sin t + \frac{3}{4}\sin 2t - \frac{1}{2}\sin 3t \right]_0^{2\pi} = 5\pi^2
\end{aligned}
$$