Introduction

The following are a selection of questions which could be asked of an interview candidate at Oxford or Cambridge. Some colleges also ask a candidate to take a short test (usually an hour long) when they come for interview. The output from these tests form a basis for discussion in the interview.

The problems are non standard in the sense they require more than just book work. They test the candidate's ability to think creatively around subjects they learn at school. Some of them are decidedly tricky!

Questions

1. Sketch the graph of $y = \sin\frac{1}{x}$.

2. Differentiate x^x with respect to x.

3. What is the value of i^i?

4. Comment on the following argument: $\displaystyle\int_0^3 \frac{1}{(1-x)^2}dx = \left[\frac{1}{1-x}\right]_0^3 = -\frac{3}{2}$.

5. Show that $\displaystyle\int_0^1 \frac{1}{1+x}dx = \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

6. Evaluate $\arctan(x) + \arctan\left(\frac{1}{x}\right)$, where x is any real number.

7. How many games are played in a knockout competition involving 2^N teams?

   How many additional games are needed to determine the second best team?

8. A washing machine has an automatic rinsing cycle in which clean water is admitted to the machine at a rate F, which can be varied by adjusting the inlet tap. When the machine contains a volume V of water, a pump is switched on and, although water continues to enter, it is pumped out at a constant rate $P(> F)$ until the machine is empty, at which point the pump switches off and the cycle starts again. Find the minimum period for the cycle. How could you determine, by observing a single cycle, whether to increase or decrease F in order to make the cycle faster?

9. Let L_1 and L_2 be two lines in the plane, with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Suppose that they intersect at an acute angle θ. Show that

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

10. Calculate $\displaystyle\int_0^\pi (x\sin x)^2 dx$

11. Of the numbers $1, 2, 3, \ldots, 6000$, how many are *not* multiples of 2, 3 or 5?

12. There is a pile of 129 coins on a table, all unbiased except for one which has heads on both sides. David chooses a coin at random and tosses it eight times. The coin comes up heads every time. What is the probability that it will come up heads the ninth time as well?

13. A box is held on the side of a hill and given a kick down the hill. The hill makes an angle θ to the horizontal, and the coefficient of friction between the packing case and the ground is μ. What relationship between μ and θ guarantees that the packing case eventually comes to rest? Let gravitational acceleration be g. If the relationship above is satisfied, what must the initial speed of the box be to ensure that the distance it goes before stopping is d?

14. Let $\binom{n}{r}$ stand for the number of subsets of size r taken from a set of size n. (This is the number of ways of choosing r objects from n if the order does not matter.) Every subset of the set $1, 2, \ldots, n$ either contains the element 1 or it doesn't. By considering these two possibilities, show that

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$

By using a similar method, or otherwise, prove that

$$\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}.$$

15. Twenty balls are placed in an urn. Five are red, five green, five yellow and five blue. Three balls are drawn from the urn at random without replacement. Write down expressions for the probabilities of the following events. (You need not calculate their numerical values.)

   a) Exactly one of the balls drawn is red.
   b) The three balls drawn have different colours.
   c) The number of blue balls drawn is strictly greater than the number of yellow balls drawn.

16. Let M be a large real number. Explain briefly why there must be exactly one root ω of the equation $Mx = e^x$ with $\omega > 1$. Why is $\log M$ a reasonable approximation to ω? Write $\omega = \log M + y$. Can you give an approximation to y, and hence improve on $\log M$ as an approximation to ω?