SOLUTIONS FOR ADMISSIONS TEST IN

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 3 NOVEMBER 2010

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. The line $y = kx$ intersects the parabola $y = (x-1)^2$ when the equation

$$(x-1)^2 = kx \iff x^2 - (k+2)\,x + 1 = 0$$

has real solutions. This quadratic equation has discrimant $(k+2)^2 - 4$ which is nonnegative when

$$k+2 \geqslant 2, \quad \text{i.e. } k \geqslant 0 \qquad \text{or} \qquad k+2 \leqslant -2, \quad \text{i.e. } k \leqslant -4.$$

The **answer is (c)**.

B. The odd terms in the sequence

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots,$$

from amongst the first $2n$ terms, are $1, 2, 4, \ldots, 2^{n-1}$ and the relevant even terms are their reciprocals. So, recognising these as geometric series, we need to sum

$$\begin{aligned} & \left(1+2+4+\ldots+2^{n-1}\right) + \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}\right) \\ = \; & \frac{1\,(2^n - 1)}{(2-1)} + \frac{1\,(2^{-n} - 1)}{(1/2 - 1)} \\ = \; & (2^n - 1) + \left(2 - 2^{1-n}\right) \\ = \; & 2^n + 1 - 2^{1-n}. \end{aligned}$$

The **answer is (a)**.

C. If x solves the equation

$$\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$$

then $\cos x \neq 0$, so that we can divide by $\cos^2 x$ to find

$$\tan^2 x + 3\tan x + 2 = 0.$$

This factorises as

$$(\tan x + 2)\,(\tan x + 1) = 0.$$

The equations $\tan x = -2$ and $\tan x = -1$ each have one solution in the range $-\pi/2 < x < 0$ and one solution in the range $\pi/2 < x < \pi$. So, overall, the original equation has 4 solutions in the range $0 \leqslant x < 2\pi$. The **answer is (d).**

D. The function $y = \sin^2 \sqrt{x}$ only takes nonnegative values which discounts (a). The minimum and maximum values of y are 0 and 1 which discounts (c). Further the zeros of y are at $x = n^2\pi^2$, which do not occur at regular intervals, and this discounts (d). The **answer is (b).**

E. Note that $\log_2 3 > 1 > \log_3 2$. Also note

$$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}.$$

We are faced with decided whether or not $\log_2 3$ and $\log_5 10$ are bigger than $3/2$. Well,

$$\log_5 10 < 3/2 \iff 10 < 5^{3/2} \iff 100 < 125$$

and

$$\log_2 3 > 3/2 \iff 3 > 2^{3/2} \iff 9 > 8.$$

So we've shown $\log_2 3 > \log_4 8 > \log_5 10 > \log_3 2$ and in particular the **answer is (a).**

F. The function $y = f(x)$ is linear on the intervals

$$0 \leqslant x \leqslant \frac{1}{3}, \qquad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}, \qquad \frac{1}{2} \leqslant x \leqslant \frac{3}{4}, \qquad \frac{3}{4} \leqslant x \leqslant 1.$$

If we apply the trapezium rule to estimate the area under the graph, by sampling the function at the values $x = k/n$ where $0 \leqslant k \leqslant n$ then we will make an overestimate unless the values $0, 1/3, 1/2, 3/4, 1$ appear amongst the values k/n. This means that the lengths of the intervals $1/3, 1/2$ and $1/4$ all need to be multiples of $1/n$ or put another way that $3, 2$ and 4 all need to be factors of n. So **the answer is (d).**

G. The function f satisfies $f(1) = 1$ and also the rules

$$f(2n) = 2f(n), \qquad f(2n+1) = 4f(n).$$

The value 16 can be achieved by applying the first rule 4 times *or* by applying the first rule twice and the second rule once *or* by applying the second rule twice. However – for the second possibility – it matters what order the rules are applied. So we see the possibilities are:

$$\begin{aligned}
f(16) &= 2f(8) = 4f(4) = 8f(2) = 16f(1) = 16, &&\text{[first rule four times]}\\
f(9) &= 4f(4) = 8f(2) = 16f(1) = 16, &&\text{[second rule, first rule, first rule]}\\
f(10) &= 2f(5) = 8f(2) = 16f(1) = 16, &&\text{[second rule, first rule, first rule]}\\
f(12) &= 2f(6) = 4f(3) = 16f(1) = 16, &&\text{[second rule, first rule, first rule]}\\
f(7) &= 4f(3) = 16f(1) = 16, &&\text{[second rule twice]}
\end{aligned}$$

There are 5 possible solutions ait matters and the **answer is (c).**

H. Consider the equation

$$(x-1)(x-2)(x-3)\times\cdots\times(x-n)=k.$$

where n is positive integer n and k is a real number.

- If $n=3$ then we have a cubic function in x which we know (from the possible shapes of cubic graphs) achieves all values k. This discounts (a).
- If n is even, for example if $n=2$, we know that the graph will have a minimum value and not attain all negative values of k. This discounts (b).
- If $n=2$ and $k=-1/4$ (the minimum value of the function on the LHS) then we see that the equation has a repeated root $x=3/2$. This discounts (d).

Hence **the answer is (c)** by a process of elimination.

Alternatively we might have argued positively to see that (c) is indeed the correct answer. If

$$f(x)=(x-1)(x-2)(x-3)\times\cdots\times(x-n)$$

then we see that $f(n)=0$. As we increase x then $f(x)$ increases as each of its factors is positive and increasing. Thus, as we keep increasing x, every positive value of k will be achieved.

I. We have

$$I(a)=\int_0^a\left(4-2^{x^2}\right)\,\mathrm{d}x$$

where $a\geqslant 0$. If one considers the graph $y=4-2^{x^2}$ then we see that $y>0$ for $0<x<\sqrt{2}$ and that $y<0$ for $\sqrt{2}<x$. We see that $I(a)$ is increasing for $0<a<\sqrt{2}$ with $I(a)$ recording ever larger amounts of the area below the graph $y=4-2^{x^2}$ and which is above the x-axis. $I(a)$ reaches a maximum at $I(a)$ and is decreasing after that as negative contributions are recorded from the (signed) area where the graph $y=4-2^{x^2}$ has moved under the x-axis. So the **answer is (b).**

J. If we consider the inequality

$$a^x>c\,b^y$$

where a,b,c are positive numbers we see:

- when $a>1$ that for any fixed y the inequality will become true for suitably large values of x. This discounts (a).
- when $b<1$ that for any fixed x the inequality will become true for suitably large values of y. This discounts (b) and (c).

Hence the **answer is (d)**. Alternatively we could rewrite the inequality as

$$-x\log a+y\log b<-\log c.$$

We are now asking that the number of integer pairs (x,y) on one side of a line be finite. Thinking diagrammatically we can see that this will happen only if $-\log a>0$ and $\log b>0$ which lead to the same conclusions $a<1$ and $b>1$.

2. (i) If $a\sqrt{2}+b=c\sqrt{3}$ then squaring both sides of the equation gives

$$2a^2+b^2+2ab\sqrt{2}=3c^2.$$

If $ab\neq 0$ then

$$\sqrt{2}=\frac{3c^2-2a^2-b^2}{2ab}$$

is rational – a contradiction and so $a=0$ or $b=0$. If $a=0$ then we have $\sqrt{3}=b/c$ unless $b=c=0$; if $b=0$ then we have $\sqrt{2/3}=c/a$ unless $c=a=0$.

(ii) We have that the square of the distance from (m,n) to $\left(\sqrt{2},\sqrt{3}\right)$ equals the square of the distance from (M,N) to $\left(\sqrt{2},\sqrt{3}\right)$, or put algebraically

$$\left(m-\sqrt{2}\right)^2+\left(n-\sqrt{3}\right)^2=\left(M-\sqrt{2}\right)^2+\left(N-\sqrt{3}\right)^2. \tag{1}$$

Rearranging this gives

$$2\left(M-m\right)\sqrt{2}+\left(m^2+n^2-M^2-N^2\right)=2\left(n-N\right)\sqrt{3}.$$

By part (i) we have that $2\left(M-m\right)=0=2\left(n-N\right)$ and hence $M=m$ and $N=n$.

(iii) If a particular point (x,y) is within distance r of $\left(\frac{1}{2},\frac{1}{2}\right)$ then so will its reflection in the $x=\frac{1}{2}$ line, the $y=\frac{1}{2}$ line and in both lines as these are diameters of the circle. The coordinates of the three points are integers also. Precisely these are the points

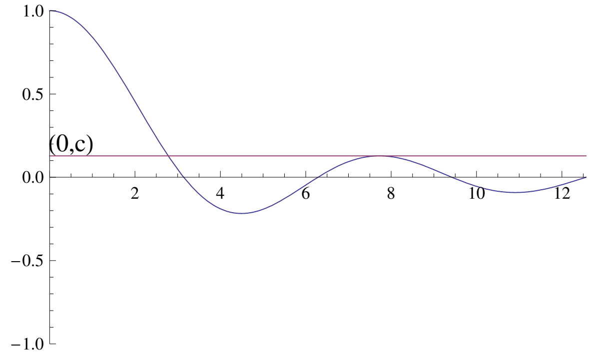
$$\left(1-x,y\right), \qquad \left(x,1-y\right), \qquad \left(1-x,1-y\right).$$

As the lattice points within the circle can be divided into sets of four like above then $N\left(\frac{1}{2},\frac{1}{2},r\right)$ is a multiple of 4.

(iv) As r increases then the circle (and its boundary) consumes lattice points. But because no two lattice points are equidistant from $\left(\sqrt{2},\sqrt{3}\right)$ – as shown in part (ii) – then the lattice points are consumed one at a time and all positive integers are achieved by $N\left(\sqrt{2},\sqrt{3},r\right)$.

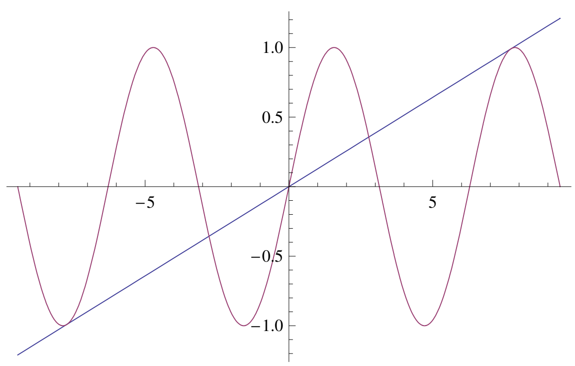
3.(i) Let x denote the angle substended by the two sides of length 1. The area of the smaller triangle is $\frac{1}{2}\sin x$, of the sector is $\frac{1}{2}x$ and of the larger triangle is $\frac{1}{2}\tan x$. So $\sin x < x < \tan x$. Multiplying the second inequality by $\cos x > 0$ (for $0 < x < \frac{\pi}{2}$) we have $x\cos x < \sin x < x$.

(ii)



We have $\cos x < \frac{\sin x}{x} < 1$ for small values of x. As $\cos x \approx 1$ for small values of x then $\sin x/x \approx 1$ for small values of 1.

(iii) The line should be drawn so that it passes through the origin and is tangential to the second hump above the y-axis and likewise tangential in the third quadrant.



(iv) The line $y = c$ should be tangential with the second positive hump of the $y = \sin x/x$ graph.

(v) As X is a solution of $\sin x = cx$ then we have $\sin X = cX$. But $y = cx$ is also tangential to $y = \sin x$ when $x = X$ and so the gradients also agree, i.e. $\cos X = c$. Eliminating c and rearranging we get $\tan X = X$. Other may know the quotient rule and show

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2} \implies \text{at } X \text{ we have } X\cos X = \sin X \implies \tan X = X.$$

4. (i) $\tan\theta = 2h$.

(ii) The point $(1, 2h)$ lies within C when $1 + 4h^2 < 4 \implies h < \sqrt{3}/2$.

(iii) Say the line connecting $(3, 0)$ and $(1, 2h)$ has equation $y = mx + c$. Then we have $3m + c = 0$ and $m + c = 2h$ giving $m = -h$ and $c = 3h$. So the line has equation

$$y = h\,(3 - x)\,.$$

This will be tangential to $x^2 + y^2 = 4$ when

$$x^2 + h^2\,(3 - x)^2 = 4$$

has a repeated root (i.e. a zero discriminant). The equation rearranges to

$$\left(h^2 + 1\right) x^2 - 6h^2 x + \left(9h^2 - 4\right) = 0.$$

The discriminant is zero when

$$36h^4 = 4\left(h^2 + 1\right)\left(9h^2 - 4\right) \implies 9h^4 = 9h^4 + 5h^2 - 4 \implies h^2 = 4/5$$

As $h > 0$ then $h = 2/\sqrt{5}$.

(iv) When $h > 2/\sqrt{5}$ then the region inside both C and T is simply a sector subtending an angle of θ at the centre of C. So the area is

$$1/2 \times (2)^2 \times \theta = 2\theta.$$

(iv) Let $h = 6/7$. The equation of the line from (i) is $y = \frac{6}{7}\,(3 - x)$. Note

$$\frac{6}{7}\left(3 - \frac{8}{5}\right) = \frac{6}{7} \times \frac{7}{5} = \frac{6}{5}$$

and so the point $(8/5, 6/5)$ does indeed lie on the line. It also lies on the circle C as

$$\left(\frac{8}{5}\right)^2 + \left(\frac{6}{5}\right)^2 = \frac{64 + 36}{25} = \frac{100}{25} = 4.$$

As the line intersects the circle, we are in the situation where the region inside both C and T comprises a sector (say subtending an angle α at 0) and a triangle. The sector again has area 2α where

$$\tan\alpha = \frac{6/5}{8/5} = \frac{3}{4}.$$

The triangle has vertices $(0, 0)\,, (8/5, 6/5)$ and $(1, 12/7)\,$. So using the given formula, its area is

$$\frac{1}{2}\left|\frac{8}{5} \times \frac{12}{7} - \frac{6}{5}\right| = \frac{1}{2} \times \frac{6}{5}\left|\frac{16}{7} - 1\right| = \frac{3}{5} \times \frac{9}{7} = \frac{27}{35}.$$

So the total area of the region inside both C and T is

$$\frac{27}{35} + 2\alpha.$$

5. (a) Let's consider dates of the form $d_1d_2/m_1m_2/20y_3y_4$. Clearly $m_1 = 1$ (to avoid repetition of 0). But then $m_2 = 0, 1$ or 2 each of which would be a repetition. Hence there are no such dates.

(b) As there are no such dates this century, let's consider dates of the form $d_1d_2/m_1m_2/19y_3y_4$. The last possible year is 1987; the last possible month is 06; and the last possible day is 25. This gives the date 25/06/1987.

(c) As there are no such dates this century, let's consider dates of the form $d_1d_2/m_1m_2/21y_3y_4$. Now $m_1 = 0$ (to avoid repetitions). Then $d_1 = 3$ (to avoid repetitions). But that leaves no possible value for d_1. Clearly there is no such date of the form $d_1d_2/m_1m_2/22y_3y_4$. For dates of the form $d_1d_2/m_1m_2/23y_3y_4$: if $m_1 = 1$ then $m_2 = 0$ and there is no possibility left for d_1. So $m_1 = 0$ and $d_1 = 1$. Of the dates $1d_2/0m_2/23y_3y_4$ the earliest possible year is 2345, the earliest possible month is 06, and the earliest possible day is 17. This gives the date 17/06/2345.

(d) Let's consider dates of the form $d_1d_2/m_1m_2/19y_3y_4$. Clearly $m_1 = 0$. If $d_1 = 3$, then $d_2 = 0$ or 1, either of which would be a repetition. Hence $d_1 = 2$. We therefore have dates of the form $2d_2/0m_2/19y_3y_4$. The remaining spaces can be filled with arbitrary distinct values from $3, 4, 5, 6, 7, 8$, giving $6 \times 5 \times 4 \times 3 = 360$ possibilities; each such possibility is a valid date.

6. (i) (a) The 6 people splits into 3 pairs sat opposite one another. For consistency each pair has to be both lying or both telling the truth. Hence there are $2^3 = 8$ possible ways in which the statements can be made.

(i) (b) If P_1 is telling the truth then P_6 is lying and P_5 is telling the truth, etc. So the only ways that the statements can be made is as $SLSLSL$ or $LSLSLS$. That is there are two ways.

(ii) (a) If a partcular person is a saint we have a LSS situation, counting left-to-right the saint and neighbours; if not then the possibilities are LLL, SLS, SLL. If we had a LSS situation the neighbour would have to be in a $SS?$ situation, but none such is permitted. Likewise SLS is impossible as there is no permitted $LS?$. And SLL is impossible as it only propagates by LLL and there is no means to complete the circle. The only possibility is $LL \ldots L$ (n times).

(ii) (b) If a person is a saint we are in a SSS or LSL situation; if not we are in a LLS or SLL situation. If we had a SSS situation this could only consistently propagate as $SSS \ldots S$ and no-one would be lying. But a LLS situation can propagate to a LSL, then to a SLL, then to LLS etc. etc. So the situation around the table can be

$$LLSLLSLLS \ldots LLS,$$

that is any number of repeats of LLS (or equivalently of LSL or SLL), provided of course that n is a multiple of 3.

7. (i) With $h = 4$ and $m = 2$ the valid games are MM, MCM, $MCCM$, CMM, $CMCM$.

(ii) When $m = h - 1$ then the mouse is only one stop away from its hole and the game will end with a single command M. Hence the possible games are $M, CM, C^2M, C^3M, \ldots, C^{h-2}M$ making $h - 1$ games in all.

(iii) If $m = 1$ then the first move has to be made by the mouse, otherwise the cat would catch it. Once the mouse has moved to the second position the remainder of the game is identical to a game where $m = 2$. So every game with $m = 1$ is of the form MG where G is a uniquely specified game with $m = 2$. Hence $g(h, 2) = g(h, 1)$.

(iv) If the game begins M then we have $G = MG'$ where G' is a game with the mouse starting at $m + 1$ and G' uniquely determined.

If the game begins C then $G = CH$. Note that H is not a valid game of Cat and Mouse, as the cat starts at 1 (rather than 0) and the mouse at m. But there will be $g(h - 1, m - 1)$ possible Hs as these H are simply a translation to the right by 1 of a game where the hole is at $h - 1$, the mouse starts at $m - 1$ and the cat at 0.

Every game has to start C or M and so $g(h, m) = g(h, m + 1) + g(h - 1, m - 1)$.

(v) The table needs to be filled in along the lines of

$$
\begin{aligned}
g(3,1) &= g(3,2) = 2; \\
g(4,1) &= g(4,2) = g(3,1) + g(4,3) = 2 + 3 = 5; \\
g(5,3) &= g(4,2) + g(5,4) = 5 + 4 = 9; \quad g(5,1) = g(5,2) = g(4,1) + g(5,3) = 5 + 9 = 14; \\
g(6,4) &= g(5,3) + g(6,5) = 9 + 5 = 14; \\
g(6,3) &= g(5,2) + g(6,4) = 14 + 14 = 28; \quad g(6,1) = g(6,2) = g(5,1) + g(6,3) = 14 + 28 = 42.
\end{aligned}
$$

A completed table is shown below.

m						
5					5	
4				4	14	
3			3	9	28	
2		2	5	14	42	
1	1	2	5	14	42	
	2	3	4	5	6	h