1.	**D**
2.	**D**
3.	**A**
4.	**D**
5.	**B**
6.	**C**
7.	**D**
8.	**C**
9.	**B**
10.	**C**
11.	**C**
12.	**D**
13.	**D**
14.	**C**
15.	**B**
16.	**E**
17.	**A**
18.	**E**
19.	**B**
20.	**B**
21.	**C**
22.	**A**
23.	**B**
24.	**B**
25.	**C**

UKMT

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**
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The Actuarial Profession
making financial sense of the future

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 8 NOVEMBER 2011

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **D** Every integer is divisible by 1; 2012 is divisible by 2 since it is even; 2013 is divisible by 3 since its digits total to a multiple of 3; and 2015 is divisible by 5 since its last digit is 5. However, 2014 is not divisible by 4 because 14 is not.

2. **D** After the first spill, $\frac{1}{3}$ of the water remains.
After the second spill, $\frac{3}{5} \times \frac{1}{3}$ of the water remains, hence $\frac{1}{5}$ of the pail had water left in it.

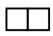
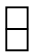
3. **A** After the n th step, *Lumber9* is at number: $\begin{cases} \frac{n+1}{2} & \text{for } n \text{ odd,} \\ \frac{-n}{2} & \text{for } n \text{ even.} \end{cases}$
Hence when $n = 2011$, *Lumber9* is at number $\frac{2011+1}{2} = 1006$.

4. **D** Since $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243, \ldots$ we see that the final digits cycle through the four numbers 3, 9, 7, 1. As $2011 = 502 \times 4 + 3$, the last digit of 3^{2011} is 7.

5. **B** As the sum of the angles in a triangle is 180° and all four angles in a rectangle are 90°, the sum of the two marked angles in the triangle is $180° - 90° = 90°$.
Each interior angle of a regular hexagon is 120° and the sum of the angles in a quadrilateral is 360°; hence the sum of the two marked angles in the quadrilateral is $360° - 90° - (360° - 120°) = 30°$.
Hence the sum of the four marked angles is $90° + 30° = 120°$.

6. **C** Let Granny's age today be G and Gill's age today be g.
Therefore $G = 15g \ldots(1)$ and $G + 4 = (g + 4)^2 \ldots(2)$.
Substituting (1) into (2) gives $15g + 4 = g^2 + 8g + 16$, hence $g^2 - 7g + 12 = 0$.
Thus $(g - 3)(g - 4) = 0$, hence $g = 3$ or 4.
As G is even and $G = 15g$, g is also even. Thus $g = 4$ and $G = 15 \times 4 = 60$.
Hence today, Granny is 56 years older than Gill.

7. **D** In order to form a triangle, x must exceed the difference between 4 and 5 and x must be less than the sum of 4 and 5, i.e. $1 < x < 9$.
Hence $x = 2, 3, 4, 5, 6, 7$ or 8. So x can have 7 different values.

8. **C** The 1×2 rectangles can appear in two different ways: A  or B .
If, in the given shape, A forms the top 1×2 rectangle then the possible different ways to fill the remaining 2×4 rectangle are, from left to right:

A, A, A, A; A, A, B, B; B, A, A, B; B, B, A, A; B, B, B, B.

If A does not form the top 1×2 rectangle, then the only possible way is to use 4B and an A. Hence there are 6 ways of dividing the given shape into 1×2 rectangles.

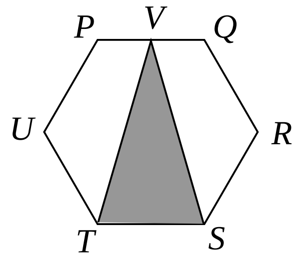
9. **B** Let the centre cube in the $3 \times 3 \times 3$ block be red. As no cubes of the same colour meet face-to-face then the 6 centre cubes on the outer faces must be yellow. All six outer faces are as shown alongside.

Y	R	Y
R	Y	R
Y	R	Y

Thus 14 faces are yellow and 13 faces are red. If the centre cube is yellow then the situation is reversed. Hence the difference between the largest number of red cubes that Sam can use and the smallest number is 1.

10. **C** The area of a triangle is $\frac{1}{2}ab \sin C$. The maximum area is attained when $\angle C = 90°$. Hence, in order to maximise the area, the triangle must be right-angled with common side lengths equal to 5. Let x be the side length of the hypotenuse, so, by Pythagoras' Theorem, $x^2 = 5^2 + 5^2 = 50$. Thus $x = 5\sqrt{2}$ is the length that should be chosen.

11. **C** Let x be the side length of the regular hexagon $PQRSTU$ and let $h = PT = QS$, the perpendicular height of triangle STV.
Thus the area of triangle STV is $\frac{1}{2}xh$ and the areas of triangles PTV and QSV are both $\frac{1}{2}\left(\frac{1}{2}xh\right) = \frac{1}{4}xh$. The perpendicular heights of triangles PTU and QRS are



$$\frac{UR - PQ}{2} = \frac{2x - x}{2} = \frac{x}{2}.$$

Hence the area of each of triangles PTU and QRS is $\frac{1}{2}h \times \frac{1}{2}x = \frac{1}{4}hx$.
Therefore the area of triangle STV is one third of the area of $PQRSTU$.

12. **D** The primorial of 7 is $2 \times 3 \times 5 \times 7 = 210$. As 8, 9 and 10 are not prime numbers, they also have a primorial of 210. The primorial of 11 is $2 \times 3 \times 5 \times 7 \times 11 = 2310$. Hence there are exactly four different whole numbers which have a primorial of 210.

13. **D** Let the centres of the starting and finishing squares in the maze have coordinates (1,4) and (4,1) respectively. Each path must pass through (2,3) and (3,2). There are two different routes from (1,4) to (2,3). The next visit is to (3,3) or (2,2).
When visiting (3,3) the next visit has to be (3,2) as (3,4), (4,3) and (4,4) cannot be visited without subsequently revisiting a square. From (2,2) the next valid visit is to (1,2), (2,1) or (3,2). From each of these points there is only one route to (3,2). Thus there are four ways of visiting (3,2). Upon visiting (3,2), the only valid route through the maze is (4,2) then (4,1).
Hence the number of different routes through the maze is $2 \times 4 = 8$.

14. **C** Let us define T_n to represent an equilateral triangle with side length n cm. Then an equilateral triangle of side length 4 cm can be divided into smaller equilateral triangles as follows:

$1 \times T_3$ and $7 \times T_1$	$4 \times T_2$	$3 \times T_2$ and $4 \times T_1$
$2 \times T_2$ and $8 \times T_1$	$1 \times T_2$ and $12 \times T_1$	$16 \times T_1$.

The number of triangles used are: 8, 4, 7, 10, 13 and 16. So it is not possible to dissect the original triangle into 12 triangles.

15. **B** If a, b are roots of $x^2 + ax + b = 0$ then $x^2 + ax + b = 0$ must be $(x - a)(x - b) = 0$. As $(x - a)(x - b) = x^2 + (-a - b)x + ab$ then $a = -a - b$ and $b = ab$. If $b = 0$ we see immediately that $a = 0$. But this is not possible as a and b are different. If $b \neq 0$ then $a = 1$ and $b = -2$. So there is just one solution pair.

16. **E** Let $QR = x$ and $RS = y$ in the rectangle $PQRS$. Hence the area of $PQRS$ is xy.
The area of triangle QRT is $\frac{1}{2}RT \times x = \frac{1}{5}xy$, hence $RT = \frac{2}{5}y$. Thus $TS = RS - RT = \frac{3}{5}y$.
The area of triangle TSU is $\frac{1}{2}SU \times \frac{3}{5}y = \frac{1}{8}xy$, hence $SU = \frac{5}{12}x$.
Therefore the area of triangle PUQ is $\frac{1}{2} \times \frac{7}{12}xy = \frac{7}{24}xy$.
Hence, as a fraction of the area of rectangle $PQRS$, the area of triangle QTU is

$$\frac{xy\left(1 - \frac{1}{5} - \frac{1}{8} - \frac{7}{24}\right)}{xy} = \frac{23}{60}.$$

17. **A** Let a, b, o and p represent the percentage of pupils liking apples, bananas, oranges and pears respectively.
As $a = 85$, there are 15% of pupils who do not like apples. As $b = 80$, $a \cap b$ is greater than or equal to $80 - (100 - 85) = 65$. As $o = 75$, $a \cap b \cap o$ is greater than or equal to $75 - (100 - 65) = 40$. Finally, as $p = 70$, $a \cap b \cap o \cap p$ is greater than or equal to $70 - (100 - 40) = 10$.
Hence the percentage of pupils who like all four fruits is at least 10%.

18. **E** Multiplying $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ throughout by $2xy$ gives $2y + 2x = xy$. Hence $xy = 2(x + y) \ldots (1)$.
But since $x^2y + xy^2 = xy(x + y)$, we can use (1) to give $xy(x + y) = 2(x + y)(x + y)$.
But $x + y = 20$, hence $x^2y + xy^2 = 2 \times 20^2 = 800$.

19. B As each square has area 1 its side length must be 1.
The external angle of the small regular octagon is $\frac{1}{8} \times 360° = 45°$.
Hence, as the sum of the angles on a straight line is 180° and the sum of the angles in a kite is 360°, the four angles in each of the eight kites (white) are: 90°, 90°, 135° and 45°.
As the light grey kites and the white kites are similar, the interior angles are the same. Two of the sides of the grey kite have length 1. Let the other sides have length a. Using the Cosine Rule twice within a light grey kite, the square of the short diagonal is $1^2 + 1^2 - 2 \times 1 \times 1 \cos 135° = a^2 + a^2 - 2a \times a \cos 45°$. Hence $2 + 2 \times 1/\sqrt{2} = 2a^2 - 2a^2 \times 1/\sqrt{2}$.
Thus $a^2 = \dfrac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}$ and so $a = \sqrt{2} + 1$.
But the area of one of the light grey kites is $2 \times \frac{1}{2}a \times 1 = a$.
Hence the area of one of the light grey kites is $\sqrt{2} + 1$.

20. B Squaring the equation $\sqrt{x} - \sqrt{11} = \sqrt{y}$ gives $x - 2\sqrt{11x} + 11 = y \ldots (1)$. You see here that $2\sqrt{11x}$ is an integer. Thus $x = 11a^2$ for some integer a. Hence in (1), $y = 11a^2 - 22a + 11 = 11(a^2 - 2a + 1)$. Thus $\dfrac{x}{y} = \left(\dfrac{a}{a-1}\right)^2$ whose maximum value, for integer a, is easily seen to be $\left(\frac{2}{1}\right)^2 = 4$.

21. C At least one of d'Artagnan and Athos is lying. One of Porthos or Aramis is telling the truth and the other is lying. So the number of liars is either two (d'Artagnan and Porthos) or three (all except Porthos).

22. A As the sum of the angles in a triangle is 180°, in triangle CBF, $\angle BFC = 90°$. As vertically opposite angles are equal $\angle DFE = \angle BFC = 90°$. As the sum of the angles on a straight line is 180°, $\angle DFB = \angle EFC = 90°$. Hence in triangle EFD, $\tan\alpha = \frac{DF}{EF}$: in triangle DFB, $\tan 10° = \frac{DF}{FB}$: in triangle BFC, $\tan 20° = \frac{FB}{FC}$ and in triangle CEF, $\tan 50° = \frac{EF}{FC}$. Thus $\tan\alpha = \frac{DF}{EF} = \frac{\tan 10° FB}{EF} = \frac{\tan 10° \tan 20° FC}{EF} = \frac{\tan 10° \tan 20°}{\tan 50°}$.

23. B $x^2 + y^2 + 2xy + 6x + 6y + 4 = (x+y)^2 + 6(x+y) + 4 = [(x+y)+3][(x+y)+3] - 5 = (x + y + 3)^2 - 5$. But $(x + y + 3)^2 \geqslant 0$ for all values of x and y. As $x + y + 3$ can be 0 for appropriate values of x, y the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$ is -5.

24. B Let the radii of the circles from smallest to largest be r_1, r_2 and r_3 respectively. Hence $16r_1 = r_3 + 2r_2 + r_1$, thus $r_3 = 15r_1 - 2r_2 \ldots (1)$. Let $r_1 + x$ be the distance from Q to the centre of the smallest circle. By similar triangles,

$$\frac{r_1}{r_1 + x} = \frac{r_2}{x + 2r_1 + r_2} = \frac{r_3}{16r_1 + r_1 + x} \ldots (2).$$

Thus $r_1(x + 2r_1 + r_2) = r_2(r_1 + x)$. Hence $r_2 = \dfrac{r_1 x + 2r_1^2}{x} \ldots (3)$. From (1) and (2) $\dfrac{r_1 x}{r_1 + x} = \dfrac{(15r_1 - 2r_2)x}{17r_1 + x}$ hence $\dfrac{r_1 x}{r_1 + x} = \dfrac{15r_1 x - 2(r_1 x + 2r_1^2)}{17r_1 + x}$. Dividing throughout by r_1 and simplifying gives $12x^2 - 8r_1 x - 4r_1^2 = 0$. Hence $(3x + r_1)(x - r_1) = 0$ so, as $r_1 > 0$, $x = r_1$. Thus $\sin\dfrac{\angle PQR}{2} = \dfrac{r_1}{r_1 + x} = \dfrac{r_1}{2r_1} = \dfrac{1}{2}$. Hence $\frac{1}{2}\angle PQR = 30°$ so $\angle PQR = 60°$.

25. C Three vertices of the smaller cube lie on edges of the larger cube, the same distance along each. Let this distance be x and let the distance between any two of these vertices be y. Hence, by Pythagoras' Theorem, $y^2 = x^2 + x^2$ and, as the side length of the smaller cube is 3, $y^2 = 3^2 + 3^2$. Thus $x = 3$ and $y = 3\sqrt{2}$.
The intersection of the cubes forms two congruent tetrahedra of base area equal to $\frac{1}{2}y^2 \sin 60° = \frac{1}{2}(3\sqrt{2})^2 \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$. Let h be the perpendicular height of the tetrahedra. Hence, using Pythagoras' Theorem twice gives $9 = h^2 + 6$, thus $h = \sqrt{3}$.
Thus the total volume of the sculpture is $4^3 + 3^3 - 2 \times \frac{1}{3} \times \frac{9\sqrt{3}}{2} \times \sqrt{3} = 91 - 9 = 82$.