1.	**E**
2.	**A**
3.	**A**
4.	**C**
5.	**A**
6.	**D**
7.	**B**
8.	**C**
9.	**C**
10.	**B**
11.	**E**
12.	**E**
13.	**D**
14.	**B**
15.	**C**
16.	**C**
17.	**E**
18.	**D**
19.	**C**
20.	**D**
21.	**B**
22.	**A**
23.	**D**
24.	**A**
25.	**E**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 9 NOVEMBER 2006

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

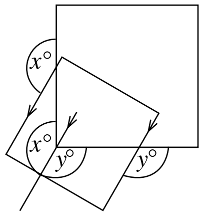
We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **E** The resulting length of the bedframe would be 80% of 2.10 m, that is 1.68 m.

2. **A** Subtracting the second equation from the first: $6x - y - (6y - x) = 21 - 14$. So $7x - 7y = 7$, that is $x - y = 1$.
(*The equations may be solved to give* $x = 4$, $y = 3$, *but it is not necessary to do this in order to find the value of* $x - y$.)

3. **A** The overlapping region of the two squares is bounded by a pentagon. Two of the interior angles of this pentagon are vertically opposite the given angles of size $x°$ and $y°$, whilst the other three interior angles are all right angles. So $x + y + 3 \times 90 = 540$, that is $x + y = 270$.
(*The diagram on the right also demonstrates that* $x + y = 270$.)



4. **C** $\sqrt{2^4 + \sqrt{3^4}} = \sqrt{16 + \sqrt{81}} = \sqrt{16 + 9} = \sqrt{25} = 5$.

5. **A** As 2006 is not a leap year, January 1st, 2007 will fall one day later in the week than January 1st, 2006, that is on a Monday. So there will be 53 Mondays in 2007 and 52 of each of the other days of the week.

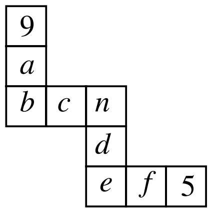
6. **D** $1 \times 2 \times (3 \oplus 4 + 5) \times (6 \times 7 + 8 + 9) = 2006$, that is $2 \times (3 \oplus 4 + 5) \times (42 + 8 + 9) = 2006$, that is $(3 \oplus 4 + 5) \times 59 = 1003$, that is $3 \oplus 4 + 5 = 17$, that is $3 \oplus 4 = 12$. So $\oplus$ should be replaced by $\times$.

7. **B** The pyramid has $2n$ edges and $n + 1$ faces, so the required difference is $2n - (n + 1)$, that is $n - 1$.

8. **C** Matt black paint reflects 3% of light, so the superblack paint reflects only 0.3% of light. Hence it absorbs 99.7% of light.

9. **C** Each spoke of the London Eye is about $\frac{1}{20}$ mile long. As 1 mile is approximately 1600 m, this means that the radius of the giant wheel is roughly 80 m. So the circumference is approximately 500 m.

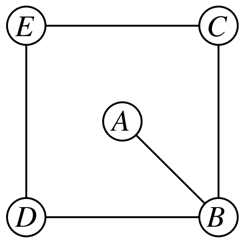
10. **B** Let a, b, c, d, e, f be the numbers in the squares shown. Then the sum of the numbers in the four lines is $1 + 2 + 3 + \ldots + 9 + b + n + e$ since each of the numbers in the corner squares appears in exactly one row and one column. So $45 + b + n + e = 4 \times 13 = 52$, that is $b + n + e = 7$. Hence b, n, e are 1, 2, 4 in some order.
If $b = 2$ then $a = 2$; if $b = 4$ then $a = 0$. Both cases are impossible, so $b = 1$ and $a = 3$.
This means that $n = 2$ or $n = 4$. However, if $n = 2$ then $c = 10$, so $n = 4$ and $c = 8$.
(*The values of the other letters are* $e = 2$, $d = 7$, $f = 6$.)



11. **E** Let the smallest of the three even numbers be n. Then the other two numbers are $n + 2$ and $n + 4$. So $4n + 2(n + 4) = 3(n + 2) + 2006$, that is $6n + 8 = 3n + 2012$, that is $n = 668$.

12. **E** It is necessary to test only $n = 2$, $n = 3$, $n = 5$ since the other two possible values are not prime. $2! + 1 = 3$, which is prime; $3! + 1 = 7$, which is prime; but $5! + 1 = 121$, which is not prime. So $n = 5$ provides the counterexample.

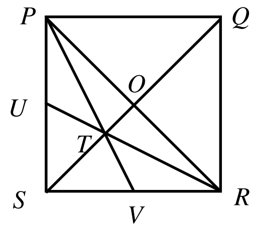
13. **D** Disc A may have any one of three colours and, for each of these, disc B may have two colours. So these two discs may be coloured in six different ways.
If discs C and D have the same colour, then they may be coloured in two different ways and, for each of these, disc E may have two colours. So the discs may be coloured in 24 different ways if C and D are the same colour. However, if discs C and D are different



colours, then C may have one of two colours, but the colours of discs D and E are then determined. So the discs may be coloured in 12 different ways if C and D are different colours. In total, therefore, the discs may be coloured in 36 different ways.

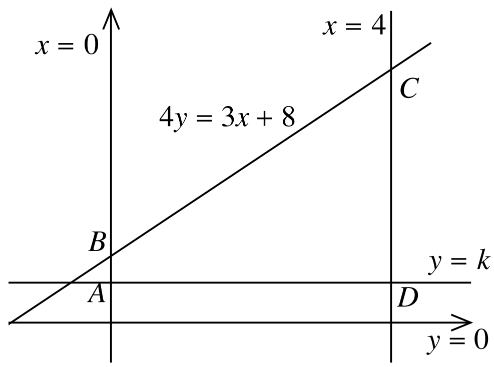
14. **B** Let Rachel and Heather have x and x^2 pennies respectively. So $x + x^2 = 100n$, where x and n are positive integers. We require, therefore, that $x(x+1) = 100n = 2^2 \times 5^2 \times n$. Now x and $x + 1$ cannot both be multiples of 5, so their product will be a multiple of 25 if and only if x or $x + 1$ is a multiple of 25. The smallest value of x which satisfies this condition is 24 which is a multiple of 4 so 24×25 is a multiple of 100. Therefore Rachel has 24 pennies, Heather has 576 pennies and, in total, they have £6.

15. **C** Let O be the centre of square $PQRS$. The medians of triangle PSR intersect at T so $OT = \frac{1}{3}OS$.
Hence the area of triangle PTR is one third of the area of triangle PSR, that is one sixth of the area of square $PQRS$. So the required fraction $= \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$.



16. **C** As $\alpha + \beta = 90°$, $\sin\alpha = \cos\beta$; $\cos\alpha = \sin\beta$. So $\sin\alpha\,\sin\beta = \sin\alpha\,\cos\alpha$; $\sin\alpha\,\cos\beta = \sin^2\alpha$; $\cos\alpha\,\sin\beta = \cos^2\alpha$; $\cos\alpha\,\cos\beta = \cos\alpha\,\sin\alpha$. As $\alpha < \beta$, $\alpha \neq 45°$. So $\sin\alpha \neq \cos\alpha$. Thus three of the four expressions have different values.

17. **E** The trapezium in question is shown as $ABCD$ in the diagram. The coordinates of its vertices are $A(0, k)$, $B(0, 2)$, $C(4, 5)$, $D(4, k)$. Using Pythagoras' Theorem: $BC = \sqrt{4^2 + 3^2} = 5$.
The perimeter of $ABCD = (2 - k) + 5 + (5 - k) + 4 = 16 - 2k$.
The area of $ABCD = 4(2 - k) + \frac{1}{2} \times 4 \times 3 = 14 - 4k$. So $16 - 2k = 14 - 4k$, that is $k = -1$.
(*In the diagram it was assumed that* $k > 0$, *although it transpires that* $k < 0$. *However, this does not affect the validity of the solution.*)

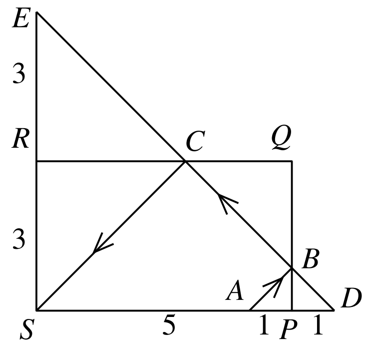


18. **D** It is not possible for all five statements to be true at the same time since if $a < b$, $a < 0$, $b < 0$ are all true then $\frac{1}{a} < \frac{1}{b}$ is not true since $\frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}$ which is negative. However, when these three statements are true, $a^2 > b^2$ is also true, so it is possible for four of the statements to be true at the same time.

19. **C** Let the length of the tunnel and the distance from the front of the train to the entrance of the tunnel when the engineer receives the warning be l and x respectively. If the engineer runs to the exit of the tunnel, he will take three times as long as he would if he ran to the entrance. So the train takes three times as long to travel a distance $x + l$ as it does to travel a distance x. Hence $l = 2x$. The train, therefore, travels a distance x in the same time that the engineer would take to travel $\frac{1}{4}l$, that is to travel $\frac{1}{2}x$. So the speed of the train is twice that of the engineer.

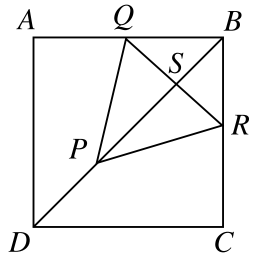
20. **D** Adding all three equations: $x + [y] + \{z\} + y + [z] + \{x\} + z + [x] + \{y\} = 4.2 + 3.6 + 2.0 = 9.8$. Now $[x] + \{x\} = x$, $[y] + \{y\} = y$, $[z] + \{z\} = z$, so $2x + 2y + 2z = 9.8$, that is $x + y + z = 4.9$. Therefore: $x + y + z - (x + [y] + \{z\}) = 4.9 - 4.2$, that is $\{y\} + [z] = 0.7$. So $[z] = 0$, $\{y\} = 0.7$.
(*It is not necessary to find the values of* x, y, z *to solve this problem, but their values may be shown to be* 1.9, 2.7, 0.3 *respectively.*)

21. **B** The route of the ball is $A \to B \to C \to S$. The diagram also shows point D, the reflection of point A in PQ, and point E, the reflection of point S in QR. As the ball bounces off a side at the same angle at which it hits that side, points D, B, C, E lie in a straight line.
Triangles DPB and DSE are similar since both are right-angled and they have a common angle at D. So $\frac{BP}{PD} = \frac{ES}{SD} = \frac{6}{7}$. Hence $BP = \frac{6}{7}$.



22. **A** The terms on the left-hand side of the equation form an arithmetic progression which has $n^3 - 5$ terms. So the sum of these terms is $\dfrac{n^3 - 5}{2}\left(\dfrac{3}{n^3} + \dfrac{n^3 - 3}{n^3}\right) = \dfrac{n^3 - 5}{2}$. Hence $n^3 - 5 = 120$, so $n = 5$.

23. **D** Let the vertices of the square be A, B, C, D and the centres of the circle and the two semicircles be P, Q, R, as shown. The midpoint of QR is S. By symmetry, P and S both lie on diagonal BD of square $ABCD$ and the whole figure is symmetrical about BD.
As P is distance 1 from both AD and DC, the length of DP is $\sqrt{2}$.
As the circles and semicircles are mutually tangent, PQR is an equilateral triangle of side 2, so the length of PS is $\sqrt{3}$. As angles QBS and BSQ are 45° and 90° respectively, triangle SBQ is isosceles, so $SB = SQ = 1$.
Hence the length of BD is $\sqrt{2} + \sqrt{3} + 1$. Now the length of the side of the square is $BD \div \sqrt{2}$ so the perimeter of the square is $4 \times (BD \div \sqrt{2})$, that is $2\sqrt{2} \times BD$.
So the perimeter is $2\sqrt{2}\,(\sqrt{2} + \sqrt{3} + 1)$, that is $4 + 2\sqrt{6} + 2\sqrt{2}$.



24. **A** Let O be the centre of the cube. Consider triangle ABO: from Pythagoras' Theorem, $OA = AB = BO = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$ cm $= \frac{1}{\sqrt{2}}$ cm. So triangle OAB is equilateral.
A similar argument may be applied to triangles OBC, OCD etc. The area of each of these equilateral triangles is $\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \sin 60°$ cm², that is $\frac{1}{8}\sqrt{3}$ cm². So the area of hexagon $ABCDEF$ is $6 \times \frac{\sqrt{3}}{8}$ cm². However, the total red area exposed by the cut is twice the area of this hexagon, that is $\frac{3\sqrt{3}}{2}$ cm².

25. **E** Let X consist of x digits, each of which is 1. So $X = \frac{10^x - 1}{9}$. Let $pX^2 + qX + r$ consist of y digits, each of which is 1. So $pX^2 + qX + r = \frac{10^y - 1}{9}$. Then $p\left(\frac{10^x - 1}{9}\right)^2 + q\left(\frac{10^x - 1}{9}\right) + r = \frac{10^y - 1}{9}$, that is $p\left(10^{2x} - 2 \times 10^x + 1\right) + 9q\left(10^x - 1\right) + 81r = 9\left(10^y - 1\right)$, that is (on dividing throughout by 10^{2x}) $p + (9q - 2p)10^{-x} + (p - 9q + 81r)10^{-2x} = 9 \times 10^{y - 2x} - 9 \times 10^{-2x}$.
We now let x tend to infinity (through integer values). The LHS of the above equation tends to p, and the second term on the right goes to 0. By continuity of the function $f(u) = 10^u = e^{u \log 10}$, we can deduce that $y - 2x$ must tend to a limit. Let this limit be L. Since $y - 2x$ is always an integer, it must actually equal L for all x sufficiently large. Passing to the limit, therefore, we obtain $p = 9 \times 10^L$. Since p is to be an integer, we must have that L (also an integer) is a non-negative integer. Substituting for p in the previous equation and simplifying leads to

$$9q - 18 \times 10^L + \left(9 \times 10^L - 9q + 81r\right)10^{-x} = -9 \times 10^{-x}.$$

Passing to the limit again leads to $q = 2 \times 10^L$ and the previous line then also gives $9 \times 10^L - 18 \times 10^L + 81r = -9$. So $r = \frac{10^L - 1}{9}$.
Possible values of (p, q, r) therefore are (9, 2, 0), (90, 20, 1), (900, 200, 11), etc. So of the values given in the question for q, only $q = 2$ is possible.
(*Observe that the three triples above correspond to* $L = 0$, $L = 1$, $L = 2$ *respectively and we note that increasing* L *by* 1 *corresponds to multiplying* $pX^2 + qX + r$ *by* 10 *and adding* 1. *As* $pX^2 + qX + r$ *consists only of* 1s, $10\left(pX^2 + qX + r\right) + 1$ *will also consist only of* 1s, *explaining why there is an infinite family of quadratics which satisfy the required condition.*)