1.	**C**
2.	**B**
3.	**D**
4.	**A**
5.	**E**
6.	**B**
7.	**E**
8.	**D**
9.	**C**
10.	**B**
11.	**A**
12.	**D**
13.	**D**
14.	**C**
15.	**E**
16.	**B**
17.	**E**
18.	**A**
19.	**A**
20.	**C**
21.	**E**
22.	**C**
23.	**B**
24.	**D**
25.	**A**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 8 NOVEMBER 2005

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **C** 2005 plus 2005 thousandths = 2005 + 2.005 = 2007.005.

2. **B** Let the five consecutive positive integers be $x - 2, x - 1, x, x + 1, x + 2$. Their sum is $5x$, so $5x = 2005$, that is $x = 401$. The five numbers are 399, 400, 401, 402, 403.

3. **D** The numbers are 1, 16, 27, 16 and 5 respectively. Their sum is 65, so their mean is 13.

4. **A** The smaller square has one ninth of the area of the larger square. So the fraction of the larger square which is shaded is half of eight ninths, that is four ninths.

5. **E** Four lengths of the indoor pool are equivalent to five lengths of the outdoor pool. So Rachel swam four ninths of the 63 days, that is 28 days, in the indoor pool.

6. **B** The longest side of any triangle is shorter than the sum of the lengths of the other two sides. This condition means that the only possible triangles having different sides of integral unit length, and having perimeters less than 13 units, have sides of length 2, 3, 4 or 2, 4, 5 or 3, 4, 5.

7. **E** The sequences have common differences of 7 and 9 respectively. The lowest common multiple of 7 and 9 is 63, so the next term after 2005 to appear in both sequences is $2005 + 63$, that is 2068.

8. **D** The first large sheet of paper will hold pages 1, 2, 19 and 20; the second will hold pages 3, 4, 17 and 18; the third will hold pages 5, 6, 15 and 16.

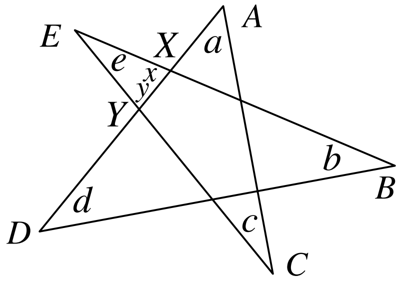
9. **C** The product is $\frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \frac{6}{5} \times \ldots \times \frac{2005}{2004} \times \frac{2006}{2005} = \frac{2006}{2} = 1003$.

10. **B** Let Sam and Pat have £x and £y respectively.
Then $y + 5 = 5(x - 5)$, that is $y = 5x - 30$. Also, $x + 5 = 5(y - 5)$, that is $x = 5y - 30$. Solving these simultaneous equations gives $x = 7.5$ and $y = 7.5$ so the friends have £15 altogether.
(*Note: from the information given, we may deduce that Sam and Pat have the same amount of money and this leads to a shorter method:* $x + 5 = 5(x - 5)$, *that is* $x = 5x - 30$, *that is* $x = 7.5$.)

11. **A** The diameter of the largest cylinder is 24cm, so the sum of the areas of the horizontal parts of the sculpture, excluding its base, is that of a circle of diameter 24cm, that is 144π cm^2. The sum of the areas of the vertical parts of the sculpture is $(2\pi \times 1 \times 2 + 2\pi \times 2 \times 2 + 2\pi \times 3 \times 2 + \ldots + 2\pi \times 12 \times 2)$ cm^2, that is 312π cm^2. So, excluding the base, the total surface area of the sculpture is 456π cm^2.

12. **D** As $\sqrt{x}$ lies between 15 and 16, x lies between 225 and 256. The multiples of 7 in this interval are 231, 238, 245 and 252.

13. **D** Let a, b, c, d, e, x and y represent the sizes in degrees of certain angles in the figure, as shown and let the points of intersection of AD with EB and EC be X and Y respectively. Angle EXY is an exterior angle of triangle XBD so $x = b + d$. Similarly, angle EYX is an exterior angle of triangle YAC so $y = a + c$. In triangle EXY, $e + x + y = 180$, so $a + b + c + d + e = 180$.

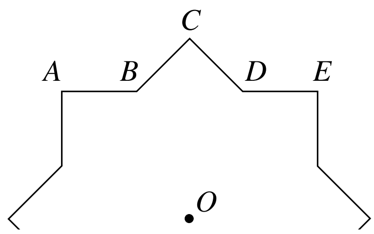


14. C When divided by 6, a whole number leaves remainder 0, 1, 2, 3, 4 or 5. So the possible remainders when a square number is divided by 6 are the remainders when 0, 1, 4, 9, 16 and 25 are divided by 6. These are 0, 1, 4, 3, 4 and 1 respectively, so a square number cannot leave remainder 2 (or remainder 5) when divided by 6.

15. E Number the four statements in order from the top. If Alice is the mother, then statements 1 and 4 are both true. If Beth is the mother, then statements 2 and 3 are both true. If Carol is the mother, then all four statements are false. If Diane is the mother, then statements 2 and 4 are both true. However, if Ella is the mother then statements 1, 2 and 3 are false and statement 4 is true, as required.

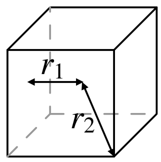
16. B Firstly, we note that of the players on the pitch at the end of the game, the goalkeeper is one of two players; the four defenders form one of five different possible combinations, as do the four midfielders, and the two forwards form one of three different possible combinations. So, if up to four substitutes were allowed, the number of different teams which could finish the game would be $2 \times 5 \times 5 \times 3$, that is 150.
From this number we must subtract the number of these teams which require four substitutions to be made. This is $1 \times 4 \times 4 \times 2$, that is 32, so the required number of teams is 118.

17. E Let A, B, C, D, E be five vertices of the star, as shown.



Then $AB = BC = CD = DE = 1$. Each exterior angle of a regular octagon is $360° \div 8$, that is $45°$, so $\angle CBD = \angle CDB = 45°$.
Hence $\angle BCD$ is a right angle and we deduce from the symmetry of the figure that each interior angle of the star is either $90°$ or $225°$. The length of BD is $\sqrt{2}$, so the area of the star is the area of a square of side $2 + \sqrt{2}$ plus the area of four congruent triangles with sides $1, 1, \sqrt{2}$.
The required area, therefore, is $(2 + \sqrt{2})^2 + 4\left(\frac{1}{2} \times 1 \times 1\right)$, that is $6 + 4\sqrt{2} + 2$, that is $8 + 4\sqrt{2}$.
Possible alternative ending: Dissect the star into 8 congruent kites such as $OBCD$. As for a rhombus, the area of a kite is half the product of its diagonals. In this case that is $\frac{1}{2}OC \times BD = \frac{1}{2}(1 + \sqrt{2}) \times \sqrt{2} = \frac{1}{2}(\sqrt{2} + 2)$. Required area is $4(\sqrt{2} + 2)$.

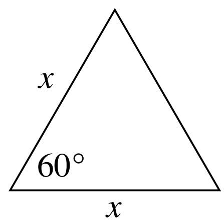
18. A Let the radii of the two spheres be r_1 and r_2, as shown.



Applying Pythagoras' Theorem: $r_2^2 = r_1^2 + r_1^2 + r_1^2$, so $r_2 = \sqrt{3}r_1$.
The ratio of the volumes of the spheres $= r_1^3 : r_2^3 = 1 : (\sqrt{3})^3$, that is $1 : 3\sqrt{3}$.

19. A When $x \geqslant 0$, $|x| = x$, so $x|x| = x^2$; when $x < 0$, $|x| = -x$, so $x|x| = -x^2$.
Only graph A has the same shape as the graph of $y = x^2$ for $x \geqslant 0$ and the same shape as the graph of $y = -x^2$ for $x < 0$.

20. C Let the length in metres of the side of a pane be x. Then the area of one pane $= \frac{1}{2} \times x \times x \times \sin 60° = \frac{\sqrt{3}}{4}x^2$. So



$$\frac{\sqrt{3}}{4}x^2 \approx \frac{6000}{3300}, \text{ that is } x^2 \approx \frac{4 \times 6000}{\sqrt{3} \times 3300}.$$

We conclude that $x^2 \approx \frac{7}{\sqrt{3}} \approx 4$.

21. E Note that $n^2 - 1$ is divisible by $n - 1$. Thus:

$$\frac{n^2 - 9}{n - 1} = \frac{n^2 - 1}{n - 1} - \frac{8}{n - 1} = n + 1 - \frac{8}{n - 1} \qquad (n \neq 1).$$

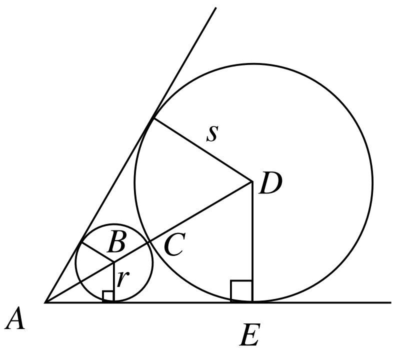
So, if n is an integer, then $\frac{n^2 - 9}{n - 1}$ is an integer if and only if $n - 1$ divides exactly into 8.
The possible values of $n - 1$ are $-8, -4, -2, -1, 1, 2, 4, 8$, so n is $-7, -3, -1, 0, 2, 3, 5, 9$.
The sum of these values is 8.
(*Note that the sum of the 8 values of n* – 1 *is clearly 0, so the sum of the 8 values of n is 8.*)

22. C There are 81 terms in the series, so, using the formula $S = \frac{1}{2}n(a + l)$ for an arithmetic series:

$$S = \frac{81}{2}(x + 20 + x + 100) = 81(x + 60).$$

Now 81 is a perfect square, so S is a perfect square if and only if $x + 60$ is a perfect square. As x is a positive integer, the smallest possible value of x is 4.

23. B We note from the symmetry of the figure that the three small circles have the same radius. Let this be r and let the radius of the large circle be s. Let A, B, C, D, E be the points shown on the diagram.



By symmetry, $\angle DAE = 30°$.
Now $\frac{DE}{AD} = \sin 30° = \frac{1}{2}$ so AD has length $2s$. Similarly, AB has length $2r$.
Since $AD = AB + BC + CD$, the length of AD is also given by $2r + r + s$. Hence $2s = 3r + s$, i.e. $s = 3r$.
Also, $\frac{DE}{AE} = \frac{s}{3/2} = \tan 30° = \frac{1}{\sqrt{3}}$ so $s = \frac{3}{2\sqrt{3}}$. Hence $r = \frac{1}{2\sqrt{3}}$.
Thus the shaded area = $\pi s^2 + 3\pi r^2 = \pi \times \frac{9}{12} + 3\pi \times \frac{1}{12} = \pi$.

24. D When $n!$ is written in full, the number of zeros at the end of the number is equal to the power of 5 when $n!$ is written as the product of prime factors, because there is at least that high a power of 2 available. For example, $12! = 1 \times 2 \times 3 \times \ldots \times 12 = 2^{10} \times 3^5 \times 5^2 \times 7 \times 11$.
This may be written as $2^8 \times 3^5 \times 7 \times 11 \times 10^2$, so 12! ends in 2 zeros, as $2^8 \times 3^5 \times 7 \times 11$ is not a multiple of 10.
We see that 24! ends in 4 zeros as 5, 10, 15 and 20 all contribute one 5 when 24! is written as the product of prime factors, but 25! ends in 6 zeros because $25 = 5 \times 5$ and hence contributes two 5s. So there is no value of n for which $n!$ ends in 5 zeros.
Similarly, there is no value of n for which $n!$ ends in 11 zeros since 49! ends in 10 zeros and 50! ends in 12 zeros. The full set of values of k less than 50 for which it is impossible to find a value of n such that $n!$ ends in k zeros is 5, 11, 17, 23, 29, 30 (since 124! ends in 28 zeros and 125! ends in 31 zeros), 36, 42, 48.

25. A

$$\frac{1}{\sqrt{2005 + \sqrt{2005^2 - 1}}} = \frac{1}{\sqrt{1003 + 1002 + \sqrt{(2005 + 1)(2005 - 1)}}}$$

$$= \frac{1}{\sqrt{(\sqrt{1003})^2 + 2\sqrt{1003}\sqrt{1002} + (\sqrt{1002})^2}} = \frac{1}{\sqrt{(\sqrt{1003} + \sqrt{1002})^2}} = \frac{1}{\sqrt{1003} + \sqrt{1002}}$$

$$= \frac{(\sqrt{1003} - \sqrt{1002})}{(\sqrt{1003} - \sqrt{1002})(\sqrt{1003} + \sqrt{1002})} = \frac{(\sqrt{1003} - \sqrt{1002})}{1003 - 1002} = \sqrt{1003} - \sqrt{1002}.$$