1.	**D**
2.	**E**
3.	**D**
4.	**D**
5.	**B**
6.	**E**
7.	**C**
8.	**C**
9.	**D**
10.	**B**
11.	**A**
12.	**C**
13.	**A**
14.	**C**
15.	**C**
16.	**E**
17.	**A**
18.	**A**
19.	**B**
20.	**E**
21.	**B**
22.	**B**
23.	**D**
24.	**E**
25.	**C**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 9 NOVEMBER 2004

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **D** For the largest of the numbers to be as large as possible, the other two numbers must be as small as possible, that is 1 and 2.

2. **E** One million minus one thousand = 1 000 000 − 1000 = 999 000. So one million million minus one thousand million = 999 000 million = 999 000 000 000.

3. **D** Let the fraction of her lolly that Milly has eaten be x. Then Molly has eaten $\frac{1}{2}x$ of her lolly. This means that the fractions they have left are $(1 - x)$ and $(1 - \frac{1}{2}x)$ respectively. So $1 - \frac{1}{2}x = 3(1 - x)$, that is $5x = 4$, giving $x = \frac{4}{5}$.

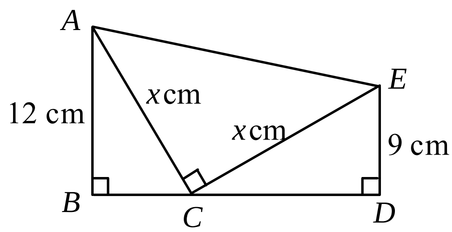
4. **D** Simon needs to buy as many batches of 5 pies as possible. He buys 400 batches of 5, giving him 2000 pies at a cost of £16. The other 4 pies cost 1p each, making a total of £16.04.

5. **B** Let the weights of squares, triangles and circles be s, t, c respectively. Then $c + 3t = 6s$; and $4c + 2t = 8s$, so $2c + t = 4s$. Hence $(c + 3t) + (2c + t) = 10s$.

6. **E** From the information given, we can conclude only that Pat is either 23 or 29.

7. **C** The original height of 29 feet 3 inches is 351 inches. So after the first bounce the ball reaches a height of 234 inches; after the second bounce the height reached is 156 inches and after the third bounce it reaches a height of 104 inches, i.e. 8 feet 8 inches.

8. **C** $\angle ACB + \angle ACE + \angle DCE = 180°$ (angles on a straight line).
So $\angle ACB = 90° - \angle DCE$.
Also, $\angle CDE + \angle CED + \angle DCE = 180°$ (angle sum of a triangle).
So $\angle CED = 90° - \angle DCE$.



Therefore $\angle ACB = \angle CED$ and we can deduce that triangles ABC and CDE are congruent since they have two pairs of equal angles and one pair of equal sides ($AC = CE$). So BC has length 9 cm and we use Pythagoras' Theorem to see that $x^2 = 12^2 + 9^2 = 225$. Hence $x = 15$.

9. **D** There are eight 2-digit numbers which satisfy the required condition (12, 23, …, 89), seven 3-digit numbers (123, 234, …, 789), six 4-digit numbers (1234, 2345, …, 6789) and one 5-digit number (12345).

10. **B** Let the small circles have radius r. Then the large circle has radius $3r$. The unshaded area is $7\pi r^2$, while the shaded area is $\pi(3r)^2 - 7\pi r^2 = 2\pi r^2$. So the required ratio is 7:2.

11. **A** In one hour the fraction of lawn which has been mowed is $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4}$. So the time taken to mow the lawn is $\frac{4}{3}$ hours, that is 1 hour 20 minutes.

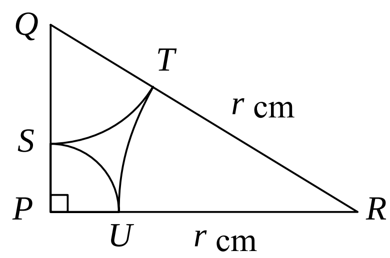
12. **C** The hexagonal face has 6 vertices and each of these must be connected to at least one other vertex in a different plane from that of the hexagonal face. So this requires at least 6 edges. Also the hexagonal face itself contributes 6 edges to the polyhedron so the polyhedron has a minimum of 12 edges. We now need to consider if such a polyhedron with 12 edges exists. It does, in the form of a pyramid on a hexagonal base, so the smallest number of edges the polyhedron could have is indeed 12.

13. **A** The last digit of 3^4 is 1, as is the last digit of 7^4 and the last digit of 9^2. So the last digit of $(3^4)^{501}$, that is of 3^{2004}, is 1. Similarly, the last digit of $(7^4)^{501}$, that is of 7^{2004}, is 1 and the last digit of $(9^2)^{1002}$, that is of 9^{2004}, is 1. Furthermore, $1^{2004} = 1$ and the last digit of 5^{2004} is 5. So the units digit of the expression is $1 + 1 + 5 + 1 + 1$, that is 9.

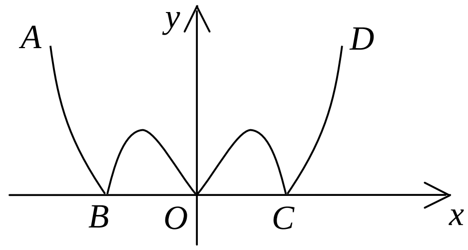
14. **C** Let the side of the cube be of length 2. Then $LM = MN = \sqrt{1^2 + 1^2} = \sqrt{2}$; $LN = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$. So LMN is an isosceles triangle with sides $\sqrt{2}$, $\sqrt{2}$, $\sqrt{6}$. Thus $\cos\angle NLM = \dfrac{\sqrt{6}/2}{\sqrt{2}} = \dfrac{\sqrt{3}}{2}$; hence $\angle NLM = 30° = \angle MNL$. So $\angle LMN = 120°$.
(Alternatively, it may be shown that L, M and N, together with the midpoints of three other edges of the cube, are the vertices of a regular hexagon. So ∠LMN may be shown to be 120°.)

15. **C** The core of the trunk occupies 81% of the volume of the trunk. Assuming that the trunk is cylindrical, this means that 81% of the cross-sectional area of the trunk is occupied by the core. Now $\sqrt{0.81} = 0.9$, so the diameter of the core is 90% of the diameter of the trunk, that is 36 cm. Hence the thickness of the bark is 4 cm ÷2, that is 2 cm.

16. **E** Let the radius of the arc with centre R be r cm.
Then $QT = (17 - r)$ cm and $PU = (15 - r)$ cm.
Now $QS = QT$ (radii of arc ST) and $PS = PU$ (radii of arc SU), so $QP = (17 - r + 15 - r)$ cm $= (32 - 2r)$ cm.
But, by Pythagoras' Theorem: $QP = \sqrt{17^2 - 15^2}$ cm $= \sqrt{(17 + 15)(17 - 15)}$ cm $= 8$ cm.
So $32 - 2r = 8$, that is $r = 12$.



17. **A** Consider the four portions AB, BO, OC and CD of the graph shown. The graph of $y = f(x)$ contains either portion AB as shown, or the reflection of this portion in the x-axis. The same is true for the other three portions, so the number of different graphs of $y = f(x)$ which would give this graph of $y = |f(x)|$ is $2 \times 2 \times 2 \times 2$ that is 16.



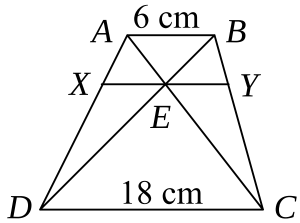
18. **A** Using the alternate segment theorem: $\angle LMN = \angle PNL = \theta°$.
Also, since $LM = LN$, then $\angle LNM = \angle LMN = \theta°$. So $\angle RNM = (180 - 2\theta)°$ (angles on a straight line).
Now $\angle LMN = \angle RNM + \angle LRP$ (exterior angle theorem), so $\angle LRP = \angle LMN - \angle RNM = \theta° - (180 - 2\theta)° = (3\theta - 180)°$.

19. **B** Multiplying the equation $S + M \times C = 64$ by S gives $S^2 + S \times M \times C = 64S$. Therefore $S^2 + 240 = 64S$, that is $(S - 60)(S - 4) = 0$. So $S = 60$ or $S = 4$. Multiplying the equation $S \times C + M = 46$ by M gives $M \times S \times C + M^2 = 46M$. Therefore $240 + M^2 = 46M$, that is $(M - 40)(M - 6) = 0$. So $M = 40$ or $M = 6$. If $M = 40$ then $S \times C = 6$, so $C = 6/60$ or $6/4$, neither of which is a whole number. If $M = 6$ then $S \times C = 40$, so $C = 40/60$ or $40/4$. Hence the only whole number solutions of the equations are $S = 4$, $M = 6$, $C = 10$.

20. **E** Expressed as the product of prime factors, $396 = 2^2 \times 3^2 \times 11$. Therefore the lowest positive integer by which it must be multiplied to make a perfect cube is $2 \times 3 \times 11^2$, that is 726.

21. **B** Note, from the information given, that $\angle PRQ = \angle RPQ = 45°$; $\angle RQS = 60°$; $\angle PQS = 30°$.
Applying the Sine Rule to ΔSRQ: $\dfrac{RS}{\sin 60°} = \dfrac{SQ}{\sin 45°}$ and to ΔSPQ: $\dfrac{SP}{\sin 30°} = \dfrac{SQ}{\sin 45°}$.
Hence $\dfrac{RS}{\sin 60°} = \dfrac{SP}{\sin 30°}$, so $RS : SP = \sin 60° : \sin 30° = \sqrt{3} : 1$.

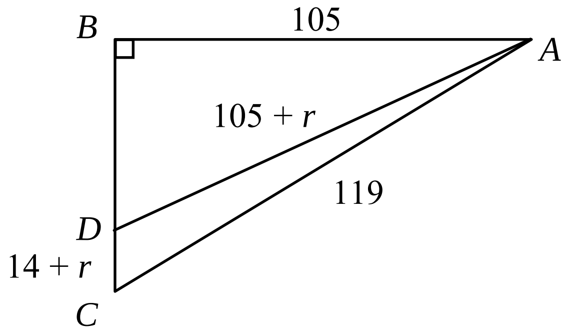
22. **B** Let the vertices of the trapezium be A, B, C, D and let AC meet BD at E, as shown. Triangles ABE and CDE are similar since $\angle ABE = \angle CDE$ (alternate angles) and $\angle AEB = \angle CED$ (vertically opposite angles). So $\frac{CE}{AE} = \frac{CD}{AB} = \frac{18}{6} = 3$.



Hence $AC = 4AE$ and, by a similar argument, $BD = 4BE$.
Triangles AXE and ADC are similar since $\angle XAE$ is the same angle as $\angle DAC$ and $\angle AXE = \angle ADC$ (corresponding angles). So $\frac{XE}{DC} = \frac{AE}{AC} = \frac{1}{4}$; hence $XE = 4.5$ cm.
Applying a similar argument to triangles BYE and BCD, we find that $YE = 4.5$ cm also. So the length XY is 9 cm.

23. **D** First note that 1, 11, 111, 1111, 11111 are not divisible by 7 but that $111111 = 15873 \times 7$. So any number in which all the digits are the same is divisible by 7 if the number of digits is a multiple of 6. Therefore the 2004-digit number 888…888 is a multiple of 7. Similarly, the 2004-digit number 222…222 is a multiple of 7, as is the 2004-digit number 222…229 since it differs from 222…222 by 7. Furthermore, the 2004-digit number 222…22n is not a multiple of 7 if n is any digit other than 2 or 9.
Now $N = 222\ldots22n \times 10^{2004} + 888\ldots888$, so if N is divisible by 7 then 222…22n is divisible by 7 and we deduce that $n = 2$ or 9.

24. **E** Let A be the centre of one of the large circles, B the point where the two large circles touch, C the centre of the small circle and D the centre of the largest circle which can be placed in the shaded region, i.e. the circle which touches all three of the given circles. Let the radius of this circle be r.
Then $AB = 105$; $AD = 105 + r$, $AC = 105 + 14 = 119$, $CD = 14 + r$.



Applying Pythagoras' Theorem to triangle ABC we find that $BC = \sqrt{119^2 - 105^2} = \sqrt{224 \times 14} = 56$. So $BD = 42 - r$.
Similarly, $(105 + r)^2 = 105^2 + (42 - r)^2$. This leads to the equation $294r = 42^2$, that is $r = 6$.

25. **C**

$$\sqrt{x + \tfrac{1}{2}\sqrt{y}} - \sqrt{x - \tfrac{1}{2}\sqrt{y}} = 1.$$

Therefore $$x + \tfrac{1}{2}\sqrt{y} - 2\sqrt{x^2 - \tfrac{1}{4}y} + x - \tfrac{1}{2}\sqrt{y} = 1,$$

that is $$2x - 1 = 2\sqrt{x^2 - \tfrac{1}{4}y}.$$

Therefore $$4x^2 - 4x + 1 = 4\left(x^2 - \tfrac{1}{4}y\right),$$

that is $$y = 4x - 1.$$

So y must be 1 less than a multiple of 4.

Of the values offered, the only possibility is 7. This gives $x = 2$, but it is necessary to check that these values are solutions of the original equation since they were derived from an argument which involved squaring equations. It is not difficult to show that $x = 2$, $y = 7$ satisfy the second equation in the above solution. We now need to confirm that $\sqrt{2 + \tfrac{1}{2}\sqrt{7}} - \sqrt{2 - \tfrac{1}{2}\sqrt{7}}$ does indeed equal 1 rather than -1, which we are able to do since it is clear that $\sqrt{2 + \tfrac{1}{2}\sqrt{7}}$ is greater than $\sqrt{2 - \tfrac{1}{2}\sqrt{7}}$.