1.	D
2.	A
3.	A
4.	C
5.	A
6.	B
7.	C
8.	E
9.	B
10.	D
11.	C
12.	C
13.	A
14.	E
15.	D
16.	A
17.	C
18.	E
19.	E
20.	B
21.	B
22.	C
23.	B
24.	A
25.	D

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 11 NOVEMBER 2003

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **D** The differences between the given years and 2003 are 498, 398, 298, 198 and 98 respectively. Of these, only 198 is a multiple of 11.

2. **A** The area of a triangle = $\frac{1}{2}$ base × perpendicular height. The four triangles have bases of equal length, since $QR = RS = ST = TU$, and the same perpendicular height, PU.

3. **A** $2 \oplus 6 = \sqrt{12 + 4} = 4$, so $(2 \oplus 6) \oplus 8 = 4 \oplus 8 = \sqrt{32 + 4} = 6$.

4. **C** Sophie, Stephanie and Sarah are all shorter than Susan. However, Susan is shorter than Sandra, so Sandra is the tallest of the five girls.

5. **A** Second prize was 18 × £4.25 − 18 × 25p = 18 × £4 = £72. So in 2000, second prize was £22 more than first prize!

6. **B** Point F must be reached directly from B, C or E. There is one route from A to B, one route from A to C, and there are three routes from A to E. So the number of possible routes is $1 + 1 + 3 = 5$.

7. **C** For ropes of the same material and same length, the weight is directly proportional to the cross-sectional area and hence to the square of the diameter of the rope. So the weight, in kg, of the second rope $\approx 2.7 \times \frac{11^2}{9^2} = 2.7 \times \frac{121}{81} = \frac{121}{30} \approx 4$.

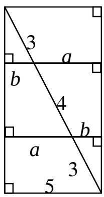
8. **E** Let the smaller number be x and the larger number y.
Then $y - x = \frac{1}{4}(y + x)$; $4y - 4x = y + x$; $3y = 5x$; hence $x : y = 3 : 5$.

9. **B** Mary's and Margaret's salaries before the pay rises were £23 100 $\times \frac{100}{110}$ and £23 100 $\times \frac{100}{105}$ respectively, i.e. £21 000 and £22 000 respectively. So the difference was £1000.

10. **D** The average speed of the balloon, in km/h, $\approx \frac{\pi \times 12750}{13\frac{1}{2} \times 24} \approx \frac{\pi \times 1000}{24} \approx \pi \times 40 \approx 120$.

11. **C** When Zerk made a beeline back to the hive, the distance she travelled is equal to the length of a space diagonal of a cube of side 1m, i.e. $\sqrt{1^2 + 1^2 + 1^2}$ m = $\sqrt{3}$ m.
So the total distance she flew = $(3 + \sqrt{3})$m.

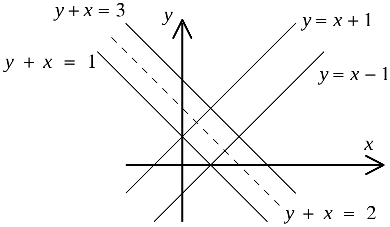
12. **C** Using similar triangles: $\frac{b}{3} = \frac{a}{7} = \frac{5}{10}$ so $a = 3\frac{1}{2}$ and $b = 1\frac{1}{2}$. Hence $a + b = 5$.



Alternatively, if a copy of the large triangle is rotated through 180° about the midpoint of its hypotenuse, then the two triangles form a rectangle and it is clear that $a + b = 5$.

13. **A** Since $x^3 - x = x(x^2 - 1) = (x - 1) \times x \times (x + 1)$, $x^3 - x$ is always the product of three consecutive whole numbers when x is a whole number. As one of these must be a multiple of 3, $x^3 - x$ will be divisible by 3. Substituting 2 for x in the expressions in B, C and E and substituting 3 for x in the expression in D results in numbers which are not divisible by 3.

14. **E** As shown in the diagram, lines A, B, C and D determine a square of side $\sqrt{2}$.



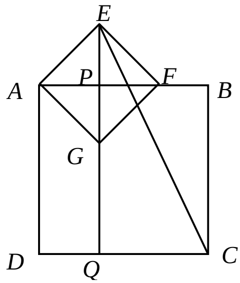
15. **D** The only square numbers which are factors of 2003^{2003} have the form 2003^{2n} for a non-negative integer n. But $2n \leqslant 2003$ so $n = 0, 1, \ldots, 1001$.

16. **A** Let the costs in pence of a peach, an orange and a melon be x, y, z respectively. We need $x - y$. We are given that $5x + 3y + 2z = 318$ and $4x + 8y + 3z = 449$. Multiplying the first by 3 and the second by 2 and subtracting gives $7x - 7y = 3 \times 318 - 2 \times 449 = 954 - 898 = 56$, so $x - y = 8$.
(*Note that as we have three unknowns, but only two equations, it is impossible to determine unique values of x, y and z. However, as has been shown, in this case it is possible to calculate the difference between the values of x and y.*)

17. **C** Let the midpoint of AF be P; as $AEFG$ is a square, this is also the midpoint of EG. Let EG produced meet CD at Q. Now $AF^2 = AE^2 + EF^2 = 4 + 4 = 8$. So $AF = \sqrt{8} = 2\sqrt{2}$ and hence $PF = PE = \sqrt{2}$.
In right-angled triangle CQE:

$$CE^2 = CQ^2 + QE^2 = \left(4 - \sqrt{2}\right)^2 + \left(4 + \sqrt{2}\right)^2$$
$$= 16 - 8\sqrt{2} + 2 + 16 + 8\sqrt{2} + 2 = 36.$$

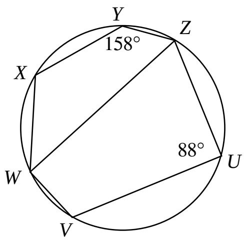
So the length of CE is 6cm.



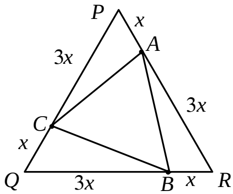
18. **E** $2^{2003} - 2^{2002} - 2^{2001} - 2^{2000} = 2^{2000}\left(2^3 - 2^2 - 2 - 1\right) = 2^{2000}(8 - 4 - 2 - 1) = 2^{2000}$.

19. **E** The graph shows that the equation linking y and x is of the form $\frac{1}{y} = m\sqrt{x} + c$, where m and c are positive constants. Hence: $y = \frac{1}{m\sqrt{x} + c}$ so $y^2 = \frac{1}{m^2x + 2mc\sqrt{x} + c^2}$.
Of the given equations, only E is of the correct form. It is obtained from the equation above when $m = c = 1$.

20. **B** The opposite angles of a cyclic quadrilateral add up to $180°$ so $\angle XWZ = 180° - 158° = 22°$; $\angle VWZ = 180° - 88° = 92°$; hence $\angle VWX = 92° + 22° = 114°$.



21. **B** Triangles BCQ, CAP and ABR are congruent since each has sides of x and $3x$ and an included angle of 60° (SAS). Consider triangle BCQ: its base, BQ, and height are $\frac{3}{4}$ and $\frac{1}{4}$ respectively of the base, QR, and height of triangle PQR. So area of triangle $BCQ = \frac{3}{16} \times$ area of triangle PQR and the area of triangle $ABC = (1 - 3 \times \frac{3}{16}) \times$ area of triangle $PQR = \frac{7}{16} \times 1 = \frac{7}{16}$.

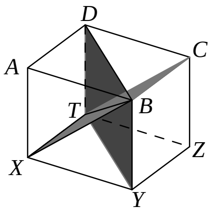


22. **C** To count the number of ways, it is necessary to have a structure. One strategy is to consider the number of £2 coins and then £1 coins; the balance can be made up with 50p coins.

Number of £2 coins	50	49	48	…	0
Maximum number of £1 coins	0	2	4	…	100
Ways	1	3	5	…	101

Total ways $= 1 + 3 + 5 + \ldots + 101 = \frac{1}{2} \times 51(1 + 101) = 51 \times 51 = 2601$.

23. **B** The piece containing corner A is a pyramid. Its base is the square $ADTX$ and its vertex is B. Let the length of the side of the cube be 1 unit: then the volume of the pyramid $= \frac{1}{3}$ base area $\times$ height $= \frac{1}{3} \times 1 \times 1 = \frac{1}{3}$. So the volume of the piece containing corner A is one third of the volume of the cube.

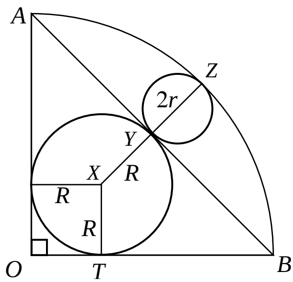


24. **A** Tangents to a circle from an exterior point are equal in length so $BT = BY = \frac{1}{2}BA = \frac{1}{2}\sqrt{2}$.

Radius $R = OT = OB - BT = 1 - \frac{1}{2}\sqrt{2}$.

Triangle OBY is isosceles with $OY = BY = \frac{1}{2}\sqrt{2}$.

Radius $OZ = OY + YZ = 1$ so $\frac{1}{2}\sqrt{2} + 2r = 1$ and hence $r = \frac{1}{2}\left(1 - \frac{1}{2}\sqrt{2}\right) = \frac{1}{2}R$.



25. **D**

$$\frac{1}{x} + \frac{2}{y} = \frac{3}{19}.$$

$$\text{i.e.} \quad 38x + 19y = 3xy.$$

$$\text{i.e.} \quad 9xy - 114x - 57y + 38 \times 19 = 38 \times 19.$$

$$\text{i.e.} \quad (3x - 19)(3y - 38) = 2 \times 19^2.$$

The factors of 2×19^2 are 1, 2, 19, 38, 361 and 722 and $3x - 19$ has to be one of these. If $3x - 19 = 1$, 19 or 361, then x is not an integer. If $3x - 19 = 2$, then $x = 7$ and $3y - 38 = 361$ giving $y = 133$. If $3x - 19 = 38$, then $x = 19$ and $3y - 38 = 19$ giving $y = 19$ as well. If $3x - 19 = 722$, then $x = 247$ and $3y - 38 = 1$ giving $y = 13$.

(*The equation* $\frac{1}{x} + \frac{2}{y} = \frac{3}{p}$ *always has exactly 3 solutions when x and y are positive integers and p is a prime greater than or equal to 5. Can you prove this*?)