1.	A
2.	B
3.	C
4.	C
5.	E
6.	C
7.	B
8.	D
9.	A
10.	E
11.	A
12.	A
13.	D
14.	E
15.	D
16.	E
17.	D
18.	B
19.	C
20.	A
21.	D
22.	B
23.	E
24.	B
25.	C

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 12 NOVEMBER 2002

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **A** A square of area 4 cm^2 has side 2 cm and therefore its perimeter is 8 cm. A square of area 1 cm^2 has side 1 cm, so its perimeter is 4 cm.

2. **B** The lines which make up a network are known as 'edges' and edges meet at points called 'nodes'. The *order* of a node is the number of edges which meet at that node. So network *A* has three nodes of order 2 and two of order 3. For a network to be traversable the number of odd-ordered nodes must be either 0 or 2. If there are no odd-ordered nodes, it is possible to trace out the network starting at any node. If there are two odd-ordered nodes, these must be the start and finish of the path which traces out the network. Network *B* has four odd-ordered nodes and therefore is not traversable.

(Is it possible for a network to have exactly one odd-ordered node?)

3. **C** The points which Peri reaches from the second day onwards are (2, 3), (3, 5), (5, 8), (8, 13) and (13, 21) respectively.

4. **C** The sum of the digits of both numbers is 45, so both are multiples of 3. Also, both numbers are odd so their product is an odd multiple of 3. Hence there is a remainder of 3 when the product is divided by 6.

5. **E** $2002 = 2 \times 7 \times 11 \times 13$; $2^2 + 7^2 + 11^2 + 13^2 = 343$.

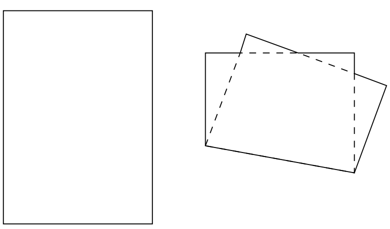
6. **C** The number of children in the village must be a multiple of both 3 and 7, i.e. it must be a multiple of 21. The only multiple of 21 which is less than 40 is 21 itself and so there are 21 children in the village. Of these, 7 can swim, 14 can ride a bicycle and 3 can do both. The number of children, therefore, who can do at least one of these activities is $7 + 14 - 3 = 18$. So 3 children can neither swim nor ride a bicycle.

7. **B** The only two-digit cubes are 27 and 64. If 1 across is 64, then 1 down must also be 64 (the only square between 61 and 69 inclusive), but there must be a different digit in each square so this is impossible. Hence 1 across is 27 and 1 down is 25. The possibilities for 2 down are now 71, 73 and 79 giving 51, 53 and 59 as possibilities for 3 across. Of these, only $53 = 2^2 + 7^2$ can be written as the sum of two squares, so 2 down is 73 and 3 across 53.

8. **D** The number of inches in a furlong is 220×36. The number of barleycorns in a furlong is $10 \times 44 \times 54$. So the number of barleycorns in one inch is $\dfrac{10 \times 44 \times 54}{220 \times 36}$, i.e. 3.

9. **A** If the trees are planted twice as far apart then the number of trees per hectare will be a quarter of its previous value.

10. **E** It is not difficult to show that the piece of paper can be folded to create polygons with 6 or 7 or 8 sides. The diagram shows that it *is* also possible to create a 9-sided polygon. When the piece of paper is folded, the fold makes up one side of the resulting polygon. In addition, each of the four corners can contribute a maximum of two sides, so the maximum possible number of sides is $1 + 2 \times 4 = 9$.



11. **A** The shaded area = $\frac{2}{3}\pi r^2 + \frac{1}{3}\left(4\pi r^2 - \pi r^2\right) = \frac{5}{3}\pi r^2$.
The unshaded area = $4\pi r^2 - \frac{5}{3}\pi r^2 = \frac{7}{3}\pi r^2$. Hence the required ratio is 5 : 7.

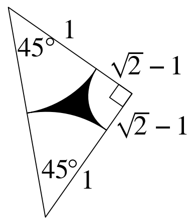
12. **A** Let yesterday's prices of a goose and an elephant be x and $99x$ respectively. Today, these prices are $\frac{11x}{10}$ and $\frac{9}{10} \times 99x$ respectively, so the required number is $\frac{9}{10} \times 99x \div \frac{11x}{10} = 81$.

13. **D** The number of different arrangements of the four cards which are dealt is $4 \times 3 \times 2 \times 1 = 24$. In only one of these will the four cards be in descending order.

14. **E** In returning to its original position, the centre of the disc moves around the circumference of a circle of diameter $2d$, i.e. a distance $2\pi d$. As the disc does not slip, its centre moves a distance πd when the disc makes one complete turn about its centre, so two complete turns are made.

15. **D** For the equation to have integer solutions, it must be possible to write $x^2 + nx - 16$ in the form $(x - \alpha)(x - \beta)$, where α and β are integers.
Therefore $x^2 + nx - 16 = x^2 - (\alpha + \beta)x + \alpha\beta$ and we require that $\alpha\beta = -16$.
The possible integer values of α, β are 1, −16; −1, 16; 2, −8; −2, 8; 4, −4 (we do not count −16, 1 as being distinct from 1, −16, for instance).
As $n = -(\alpha + \beta)$, the possible values of n are 15, −15, 6, −6 and 0.

16. **E** The triangle whose vertices are the centres of the three circles has sides of length $\sqrt{2}$, $\sqrt{2}$ and 2 and is, therefore, a right-angled isosceles triangle.
The perimeter of the shaded region is



$$2 \times \frac{1}{8} \times 2\pi + \frac{1}{4} \times 2\pi\left(\sqrt{2} - 1\right) = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}.$$

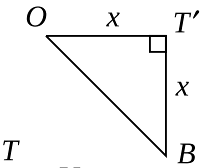
17. **D** Puzzle number 351 is in volume 5, so there is a maximum of 87 puzzles per volume (since, if there were 88, then puzzle 351 would be in volume 4). Also, puzzle number 689 is in volume 8, so there are at least 87 puzzles per volume (since, if there were 86, then puzzle 689 would be in volume 9).

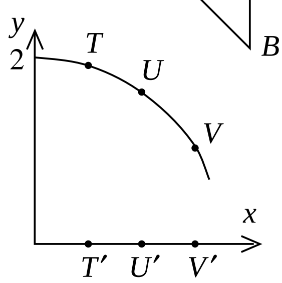
18. **B** Note that $81 = 3^4$. Therefore $\frac{81^{20}}{3^{81}} = \frac{3^{80}}{3^{81}} = \frac{1}{3}$.

19. **C** Let the point on the ground vertically below T be T', let O be the point where line DB meets the wall and let $OT' = x$.
Then, since $\angle T'OB = \angle T'BO = 45°$, $T'B = x$.
As $TB = 2$, $TT' = \sqrt{4 - x^2}$.
Hence, with respect to axes shown in the diagram, the equation of the curve on which T, U, V, … lie is

$$y = \sqrt{4 - x^2}, \text{ i.e. } x^2 + y^2 = 4,\ x \geqslant 0,\ y \geqslant 0$$

which is the equation of part of a circle of radius 2.

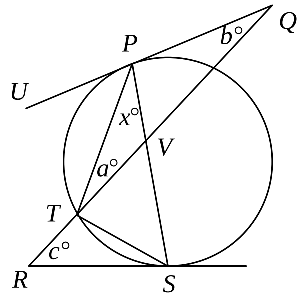




20. **A** We note that the graph of $y = \sin\left(x^2\right)$ passes through the origin and also is symmetrical about the y-axis, since $(-x)^2 = x^2$. For $-\sqrt{\pi} \leqslant x \leqslant \sqrt{\pi}$, $\sin\left(x^2\right) \geqslant 0$ and the only one of the graphs to satisfy all of these conditions is A.
[Note that the graph of $y = \sin\left(x^2\right)$ has range $-1 \leqslant y \leqslant 1$ and crosses the x-axis when $x = \pm\sqrt{n\pi}$ for all natural numbers n.]

21. **D** Note that, as on the question paper,

$$a° = 20°,\ b° = 30°,\ c° = 40°.$$



$\angle UPT = \angle PTQ + \angle PQT = 50°$ (exterior angle of ΔPQT).
$\angle TSP = \angle UPT = 50°$ (alternate segment theorem).
$\angle RST = \angle TPS = x°$ (alternate segment theorem).
$\angle STV = \angle TRS + \angle TSR = (40 + x)°$ (exterior angle of ΔRST).
So $\angle STP = (60 + x)°$. Then in ΔSTP:
$(60 + x) + x + 50 = 180$ (angle sum of a triangle).
Hence $x = 35$.

22. **B**

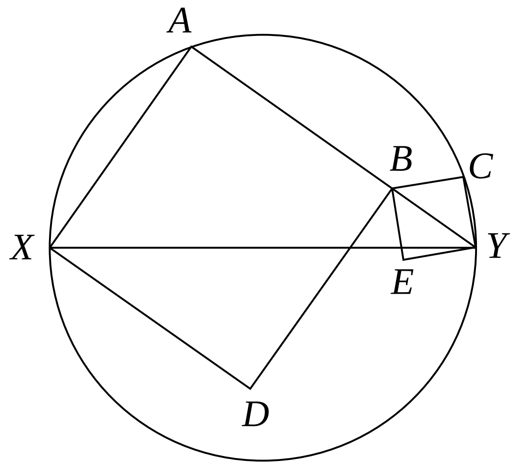
$$y = \frac{x}{x + \dfrac{x}{x+y}} = \frac{x(x+y)}{x(x+y)+x} = \frac{x+y}{x+y+1} \qquad (x \neq 0, x+y \neq 0, x+y+1 \neq 0)$$

i.e. $y(x + y + 1) = x + y$

i.e. $y^2 + xy - x = 0$

For y to be real, this quadratic equation must have real roots so $x^2 + 4x \geqslant 0$, i.e. $x(x + 4) \geqslant 0$.
This condition is satisfied when $x \leqslant -4$ or when $x \geqslant 0$.
However, $x \neq 0$ so y is real when $x \leqslant -4$ or when $x > 0$.

23. **E** As $\angle XAB = 90°$ (interior angle of a square) and $\angle XAY = 90°$ (angle in a semi-circle), ABY is a straight line.
The length of the diagonal of square $XABD = \sqrt{2}$ and the length of the diagonal of square $YCBE = AY - AB = \sqrt{3} - 1$.
Hence the required ratio is
$(\sqrt{2})^2 : (\sqrt{3} - 1)^2 = 2 : (4 - 2\sqrt{3}) = 1 : (2 - \sqrt{3})$.
(The diagram shows C on the circumference of the circle. It is left as an exercise for the reader to prove that this is the case.)



24. **B**

$$f(2008) = \frac{f(2005) - 1}{f(2005) + 1} = \frac{\dfrac{f(2002)-1}{f(2002)+1} - 1}{\dfrac{f(2002)-1}{f(2002)+1} + 1} = \frac{f(2002) - 1 - (f(2002) + 1)}{f(2002) - 1 + f(2002) + 1}$$

$$= \frac{-2}{2 \times f(2002)} = \frac{-1}{f(2002)}$$

Hence $f(2002) \times f(2008) = -1$ provided that $f(2002) \neq 0$.

25. **C** Let N have x digits, so that $x \leqslant 2002$.
When the digit 1 is placed at its end, N becomes $10N + 1$.
When 1 is placed in front of it, N becomes $10^x + N$.
Therefore: $10N + 1 = 3(10^x + N)$, i.e. $7N = 3 \times 10^x - 1$.
So we need to find which of the numbers 2, 29, 299, 2999, 29999, …. are divisible by 7.
The first such number is 299999 (corresponding to $x = 5$), giving $N = 42857$ and we check that $428571 = 3 \times 142857$.
The next such numbers correspond to $x = 11$, $x = 17$, $x = 23$, $x = 29$ and the largest number in the given range corresponds to $x = 1997$.
The number of different values of N, therefore, is $1 + (1997 - 5) \div 6 = 333$.