Junior Mathematical Olympiad 2017

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working or false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2017, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1. An amount of money is to be divided equally between a group of children. If there was 20p more than this amount, then there would be enough for each child to receive 70p. However, if each child was to receive 60p, then £2.10 would be left over.

How many children are there in the group?

Solution:

B1) Let the total amount of money be x.
Let the total number of children be c.

We know that:

$$20 + x = 70c$$
$$60c = x - 210$$

Hence,
$$x = 60c + 210$$

Now if we substitute the value of x in the first equation.

$$20 + 60c + 210 = 70c.$$

After simplifying this we get,

$$230 = 10c$$
$$\underline{\underline{c = 23.}}$$

We can check this by the original 2 statements.

$23 \times 60p = £13.80.$
So the total amount of money $= £13.80 + £2.10$
$= £15.90$

If we use this in the other equation,
$£15.90 = £16.10$
$1610 \div 70 = 23.$

Hence the total no. of children in the group is 23.

A shorter solution:

If giving everyone 60p would leave £2.10 but adding 20p means you can give every child 10p more, then this would mean £2.30 can be split into 10p for each child. That means there are £2.30 ÷ 10p = 23 children.

B2. A 3-digit integer is called a 'V-number' if the digits go 'high-low-high' – that is, if the tens digit is smaller than both the hundreds digit and the units (or 'ones') digit.
How many 3-digit 'V-numbers' are there?

Solution

B2) Let's call the tens digit of a V-number 't'

For every possible single digit value of t, there are n possibilities for the other digits.

$n = 9 - t$, so for instance if $t = 7$, there are $(9-7)$ possibilities for the other digits. In this case, those possibilities are 8 and 9.

For every value n, the number of possible ways to combine two of the digits higher than t is n^2.

t can be 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 so the total number of V-numbers can be written as:

$(9-0)^2 + (9-1)^2 + (9-2)^2 + (9-3)^2 + (9-4)^2 + (9-5)^2 + (9-6)^2 + (9-7)^2 + (9-8)^2 + (9-9)^2$

This is equal to

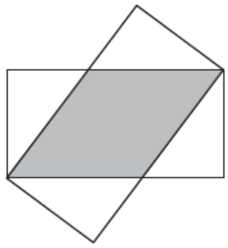
$81 + 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 + 0$

which is equal to 285

so there are 285 possible 3 digit V-numbers.

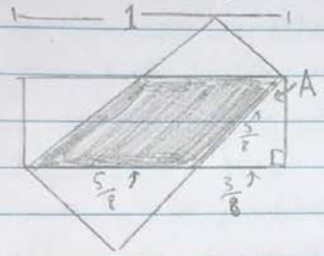
B3. Two identical rectangles overlap in such a way that a rhombus is formed, as indicated in the diagram. The area of the rhombus is five-eighths of the area of each rectangle.

What is the ratio of the length of the longer side of the rectangle to the length of the shorter side?



Solutio

- The way to find the area of a rhombus is length $\times$ height.
- The height of the rhombus is the shorter side of the ~~triangle~~ rectangle
- The way to find the area of a rectangle is also length $\times$ height.
- Because the height of the rhombus is the same as the height of the rectangle, and the area of the rhombus is $\frac{5}{8}$ the area of the rectangle, we know that all the sides of the rhombus are equal to $\frac{5}{8}$ of the longer side of the rectangle.



- This means the edge that's not part of the rhombus is $\frac{3}{8}$ the length of the longer side.
- Because all angles in a rectangle are 90°, A is a right angled triangle.
- By using Pythagoras' theorem, $a^2 + b^2 = c^2$, so $\left(\frac{3}{8}\right)^2 + b^2 = \left(\frac{5}{8}\right)^2$
- So, $\frac{9}{64} + b^2 = \frac{25}{64}$
- So, b^2 must be $\frac{16}{64}$.
- $\sqrt{\frac{16}{64}} = \frac{4}{8}$, so the shorter side of the rectangle is $\frac{4}{8} = \frac{1}{2}$ ~~smaller than~~ the length of the longer side of the rectangle.
- So, because the longer side is double the shorter side of the rectangle, the ratio between the longer side of the rectangle to the shorter side is 2:1.

B4. My uncle lives a long way away and his letters always contain puzzles. His three local teams are the Ants (A), the Bees (B), and the Cats (C), who play each other once a year.

My uncle claimed that the league table part way through the year looked like this:

	Played	Won	Drawn	Lost	Goals for	Goals against
A	1	0	0	1	4	2
B	2	1	1	0	2	2
C	2	1	0	1	3	1

When we complained that this is impossible, he admitted that every single number was wrong but he excused himself because every number was exactly '1 out'.

Find the correct table, explaining clearly how you deduced the corrections.

Solutio

B4)

	P	W	D	L	GF	GA
A	1	0	0	1	4	2
B	2	1	1	0	2	2
C	2	1	0	1	3	1

The A team must have won and drawn exactly one game as -1 games is not possible.
Therefore the A team can only have played 2 games and lost 0.

Similarly, the B team must have lost 1 game, as -1 games isn't possible.

The C team must have drawn 1 game.

The table now looks like this (circled numbers are correct)

	P	W	D	L	GF	GA
A	2	1	1	0	4	2
B	2	1	1	1	2	2
C	2	1	1	1	3	1

No team can have played 3 matches as each team plays once a year, so B must have won and drawn 0 and C must have won and lost 0, meaning both teams played once.

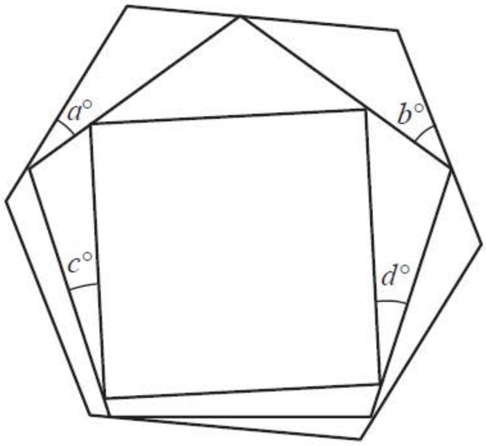
	P	W	D	L	GF	GA
A	2	1	1	0	4	2
B	1	0	0	1	2	2
C	1	0	1	0	3	1

B lost its game so must have had 1 goal for and 3 against.
C drew its game so must have had 2 goals for and against
A won 1 and drew 1 so must have had at least 2 goals against and 5 for, as it drew against C and won against B

	P	W	D	L	GF	GA
A	2	1	1	0	5	3
B	1	0	0	1	1	3
C	1	0	1	0	2	2

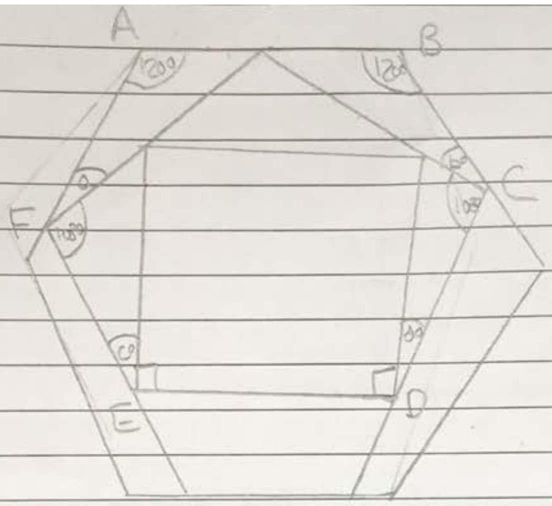
B5. The diagram shows a square whose vertices touch the sides of a regular pentagon. Each vertex of the pentagon touches a side of a regular hexagon.

Find the value of $a + b + c + d$.



Solution

B5.



In the diagram above we are asked to work out the sum of a, b, c and d. I have labelled angles based on the interior angles of a square being 90°, a pentagon 108° and a hexagon 120°.

We can see from the hexagon ABCDEF that it includes the angles a, b, c and d and some others, which are labelled. We can now construct the formula:

$a^\circ + 120^\circ + 120^\circ + b + 108^\circ + d^\circ + 90^\circ + 90^\circ + c + 108^\circ = 720^\circ$ (the total amount that angles in a hexagon sum up to).

This rearranges into: $a^\circ + b^\circ + c^\circ + d^\circ + 636^\circ = 720$

which finally leaves: $a^\circ + b^\circ + c^\circ + d^\circ = 84^\circ$.

So the sum of $a^\circ + b^\circ + c^\circ + d^\circ = 84^\circ$.

B6. The 9-digit positive integer N with digit pattern *ABCABCBBB* is divisible by every integer from 1 to 17 inclusive.

The digits A, B and C are distinct. What are the values of A, B and C?

Solution

The smallest positive integer that is a multiple of 1-17 inclusive is not 17!, as many prime factors are repeated. It is in fact, $2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 17$. This equals 12,252,240, which, of course, is not 9 digits. If we multiply it by 50 however we find a 9 digit integer. This is 612,612,000, which almost satisfies the condition, ABCABCBBB, but the second and fifth digits are not equal to the last 3. But, however, if we halve this, which is equivalent to multiplying our original 12,252,240 by 25, it becomes 306,306,000 and this number satisfies the conditions. It is a multiple of every integer 1-17 inclusive and has the digit pattern ABCABCBBB. Of course, the question asks for the values of A, B and C. A=3, B=0 and C=6.