Junior Mathematical Olympiad 2016

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough working or false starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2016, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1. In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is 30° larger than the average of the other two angles.
What is the largest possible size of an angle in this triangle?

Solution:

First, let a be the average size of the other two angles. We can work out the sum of these two angles as since there are two angles, the sum of the angles is equal to double the average. (2a)
We also know that the third angle in the triangle is equal to thirty more than the average of the other two, (a + 30)
Because the sum of all the angles in a triangle is 180°, we know that:

$$2a + a + 30 = 180$$

Simplifying this, we get:

$$3a + 30 = 180$$

If we subtract 30, we get:

$$3a = 150$$

Dividing each side by 3, we get:

$$a = 50°$$

This means one angle is equal to 80° and the other two have a sum of 100°.
To get the largest possible angle, we can split the 100° into 99° and 1°. So the largest possible size of an angle is 99°

B2. The points A, B and C are the centres of three circles. Each circle touches the other two circles, and each centre lies outside the other two circles. The sides of the triangle ABC have lengths 13 cm, 16 cm and 20 cm.
What are the radii of the three circles?

Solution:

First, let the radii of the three circles be a, b and c.

Each side of the triangle ABC is the sum of two different radii, so:

$a + b = 13$
$b + c = 16$
$c + a = 20$

Since $c + b = 16$ and $c + a = 20$, this means a is four more than b:

$b + 4 = a$

Repeating this with the other radii, we get:

$b + 7 = c$
$a + 3 = c$

This means that;

$a + a - 4 = 13$

Simplifying this, we get:

$2a - 4 = 13$

If we add four, we get:

$2a = 17$

Finally, dividing by two, we get:

$a = 8.5\,cm$

If we repeat this for the other two radii, we get:

$b = 4.5\,cm$

and:

$c = 11.5\,cm$

So the three radii are: 8.5 cm, 4.5 cm and 11.5 cm.

B3. A large cube consists of a number of identical small cubes. The number of small cubes that touch four other small cubes face-to-face is 168.

How many small cubes make up the large cube?

Solution:

Let the cube be x cubes by x cubes by x cubes.

The number of cubes in contact with 4 other cubes = 168

A cube is in contact with 4 other cubes if it is along an edge, but not a corner,

Since the number of cubes on each edge = x,

then on each edge there are (number of cubes on edge − 2 corners on edge)

cubes in contact with 4 faces = $x-2$

Since there are 12 edges, then

$$12(x-2)=168$$
$$x-2=14$$
$$x=16$$

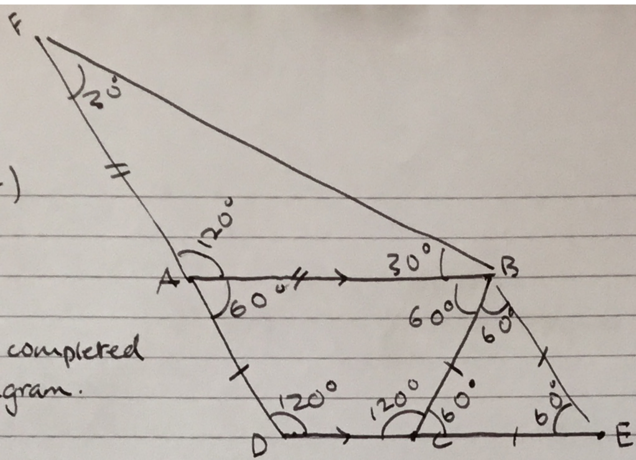
$\therefore$ the number of cubes the □ is made up of =

$16\times16\times16=\underline{4096}$

B4. In the trapezium $ABCD$, the lines AB and CD are parallel. Also $AB = 2DC$ and $DA = CB$. The line DC is extended (beyond C) to the point E so that $EC = CB = BE$. The line DA is extended (beyond A) to the point F so that $AF = BA$.
Prove that $\angle FBC = 90°$.

Solution:

B4)



The completed diagram.

In the question, it states that the lines $EC = CB = BE$. Since all 3 lines are the same lenth, the triangle BCE is an equilareral triangle, meaning each angle inside it is 60°, since angles in a ~~triagle~~ triangle add up to 180°, and $180° \div 3 = 60°$.

Since angles on a straight line add up to 180°, angle BCD is 120°

Because lines AD and BC are equal and AB and DC are parallel, the trapezium ABCD is an isoceles trapezium, meaning angle BCD is equal to ~~Angl~~ angle ADC, which means it is also 120°.

Since supplementary angles add up to 180°, angles ~~A~~ BAD and ABC are 60°, since $180° - 120° = 60°$.

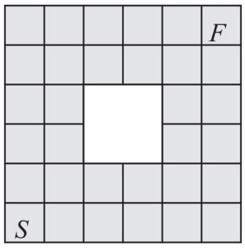
This means that angle FAB is 120°, because angles on a straight line add up to 180°.

Because lines FA and AB are the same lenth, triangle ABF is an isoceles triangle, meaning angles AFB and ABF are the same size.
Angles in a triangle add up to 180°, so by doing $(180-120) \div 2$, we get the sizes of angles AFB and ABF, which is 30°.

Finally, now we have the sizes to both angles FBA and ABC, we can find the size of angle FBC by adding 30° & 60°, which is 90°, therefore proving that angle FBC is 90°.

B5. The board shown has 32 cells, one of which is labelled S and another F. The shortest path starting at S and finishing at F involves exactly nine other cells and ten moves, where each move goes from cell to cell 'horizontally' or 'vertically' across an edge.

How many paths of this length are there from S to F?



Solution:

To have a path across the board in 9 moves, the only moves you can make are up or to the right. You can make a grid and fill every square with the number of possible routes with only up or to the right moves (the number of 'shortest path' routes)

1	6	11	16	26	52
1	5	5	5	10	26
1	4			5	16
1	3			5	11
1	2	3	4	5	6
	1	1	1	1	1

It starts off and from there every square is filled with the sum of the numbers below and to the left (the outside of the grid and 'hole' in the middle count as 0). From this diagram you can see that there is 52 'shortest path' routes that finish in the top right corner.

B6. For which values of the positive integer n is it possible to divide the first $3n$ positive integers into three groups each of which has the same sum?

Solution:

I will be doing an inductional process for even and odd n's independently.

Odd n:

If n is 1, it simply is not possible.

If n is 3:

1 2 3 4 5 6 7 8 9

Make three groups of 15.

Now suppose it is true for some odd n. For n+2 we take the first 3n and divide into three groups.

Then we have:

$3n+1$, $3n+2$, $3n+3$, $3n+4$, $3n+5$, $3n+6$

Which we pair up as shown, adding up to get three equal groups.

Q.E.D by induction

Even n:

n=2:

1 2 3 4 5 6

B6 continued:

Now, suppose it is possible for an even n.

The next even n, $n+2$, is done as follows:

We take the first $3n$ and split them into the three groups.

We have left:

$3n+1$, $3n+2$, $3n+3$, $3n+4$, $3n+5$, $3n+6$

We group them as shown and then add the $6n+7$ formed to the previous groups, $6n+7$ to each, forming three big, equal, groups.

Q.E.D by induction.

I have proven it for all odd and even n more than 1.

$\Rightarrow$ It is possible for all n more than 1.

Q.E.D.