Junior Mathematical Olympiad 2015

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough working or false starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2015, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

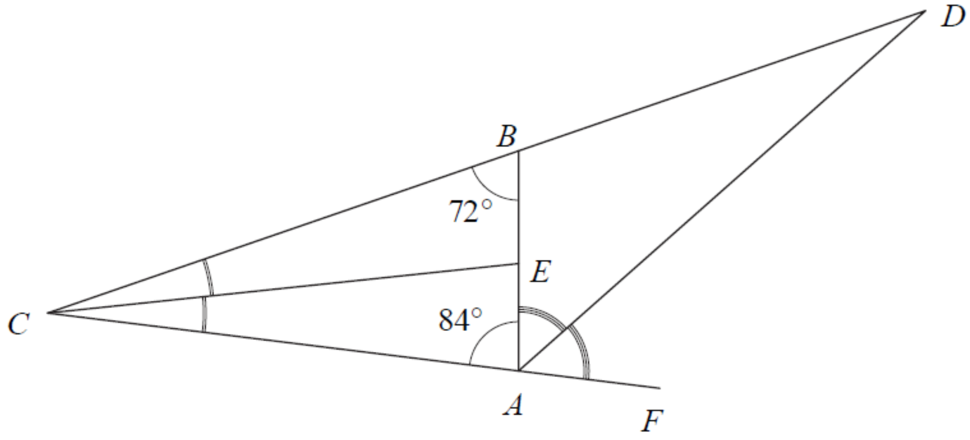
If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1. Let N be the smallest positive integer whose digits add up to 2015. What is the sum of the digits of $N + 1$?

Solution

B1. The smallest positive integer will have as few digits as possible so almost all of its digits must be 9. $2015 \div 9 = 223r8$ so there must be 223 9s and an 8 in Ns digits. For the smallest integer possible, 8 must have the highest place value and all the other digits will be 9s. When 1 is added to this number, all of the digits which were 9s will become 0s and the 8 will become a 9. Therefore, the sum of N+1s digits is 9.

B2. The diagram shows triangle ABC, in which $\angle ABC = 72^\circ$ and $\angle CAB = 84^\circ$. The point E lies on AB so that EC bisects $\angle BCA$. The point F lies on CA extended. The point D lies on CB extended so that DA bisects $\angle BAF$.



Prove that $AD = CE$.

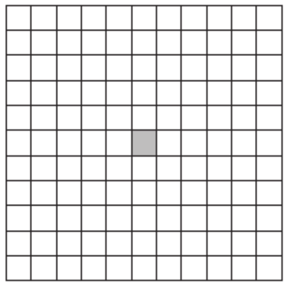
Solution

EC bisects $\angle BCA$ so $\angle BCE = \angle ECA$. $\angle BCA = 180^\circ - 72^\circ - 84^\circ = 24^\circ$ because angles in a triangle add up to 180°. This means $\angle BCE = \angle ECA = \frac{24^\circ}{2} = 12^\circ$. From this we can work out that $\angle BEC = 180^\circ - 72^\circ - 12^\circ = 96^\circ$ and $\angle CEA = 180^\circ - 12^\circ - 84^\circ = 84^\circ$. (Also $\angle CEA$ is on a straight line so $\angle CEA = 180^\circ - 96^\circ = 84^\circ$). As $\angle CEA = \angle CAE = 84^\circ$, this means ΔECA is isosceles and so lines $CE = CA$.

$\angle EAF = 180^\circ - 84^\circ = 96^\circ$. AD bisects $\angle EAF$ so $\angle EAD = \angle DAF = 96^\circ \div 2 = 48^\circ$. This means angle $CAD = 84^\circ + 48^\circ = 132^\circ$. From this we can work out that $\angle BDA = 180^\circ - 132^\circ - 24^\circ = 24^\circ$ (Using ΔCDA). This means ΔCDA is isosceles so line $AD = CA$. As $CA = CE$, this means $AD = CE$.

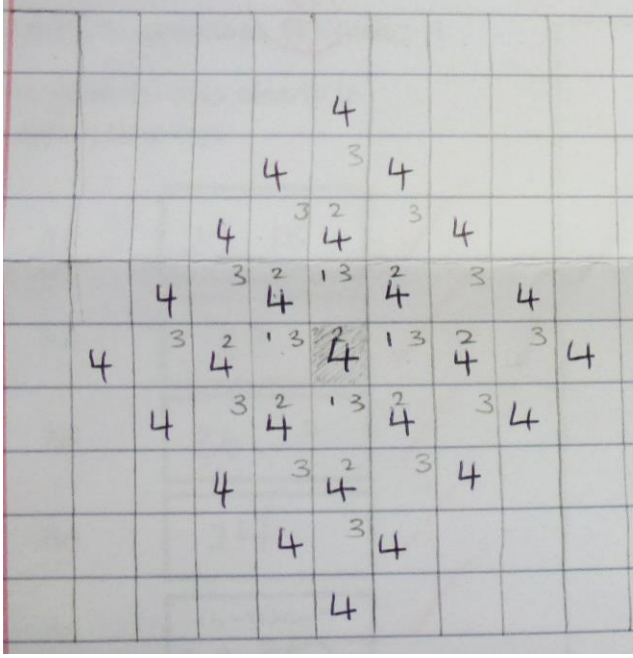
B3. Jack starts in the small square shown shaded on the grid, and makes a sequence of moves. Each move is to a neighbouring small square, where two small squares are neighbouring if they have an edge in common. He may visit a square more than once.

Jack makes four moves. In how many different small squares could Jack finish?



Solution

B3



The first move (1) can only be put on the left, right, above or below. The second move (2) can go back to the original as well as moving further out. The third move (3) can take the place of where the 1's were, and can move further out.

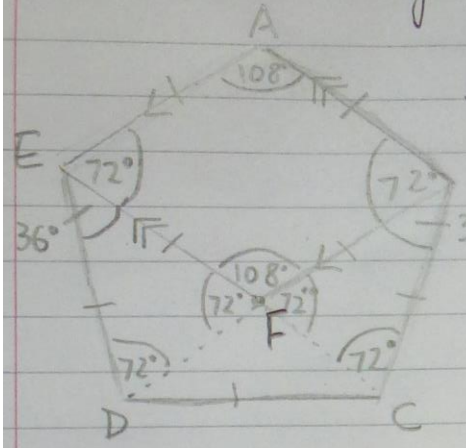
Finally, the 4^{th} move can return back to the centre, take the place of pre-existing (2)'s and move further out.

In total, there are 25 different places Jack could land on his 4^{th} move.

B4. The point F lies inside the regular pentagon $ABCDE$ so that $ABFE$ is a rhombus. Prove that EFC is a straight line.

Solution

As all sides of a rhombus are equal, and



pentagon ABCDE is regular, we know that line AE = AB = EF = BF = ED = DC = BC.

The interior angle in a regular pentagon is $\frac{180 \times (5-2)}{5} = 108°$

This means $\angle EAB = \angle EFB = 108°$.

From this we can work out $\angle AEF = \angle ABF = 360° - 108° - 108° = 72°$.

$\therefore \angle FBC = 108° - 72° = 36° = \angle FED$

Now join point F to C. $\triangle FBC$ is isosceles because line FB = BC. This means $\angle BFC = \angle BCF = (180° - 36°) \div 2 = 72°$. The pentagon ~~and~~ is regular, so the same applies to $\triangle EDF$.

$\angle ABF = \angle BFC = 72°$, this means they are alternate ('Z') angles.

$\therefore$ line AB is parallel to line FC

As ~~rhombus~~ ABFE is a rhombus, its opposite sides are parallel. This means line EF is parallel to line AB. As ~~line~~ both lines EF and FC are parallel to line AB, EFC must be a straight line.

B5. I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm.

What is the smallest square that can be made with equal numbers of each type of tile?

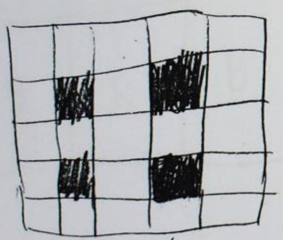
Solution

The area of one small tile and one big tile is $5\,cm^2$.
We need to make it a square number to make a square.
The lowest square number with 5 as a factor is $5^2 = 25$.

We need 5 small and 5 big tiles to fill it.

But no matter how hard we try we only can fit 4 big tiles in a 5×5 grid.

Why? Because every 2×2 tile will overlap one of the black squares shown below:

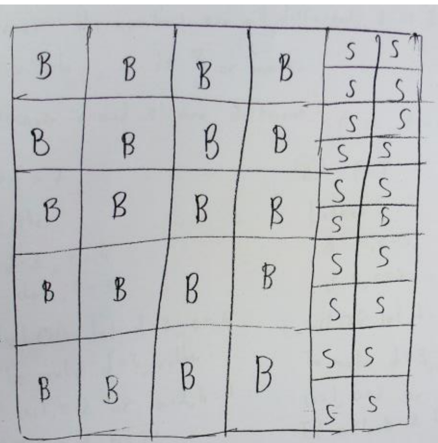


Because there are only four black squares, we can only fit four 2×2 tiles, one for each square.

So filling in a 5×5 is impossible.

The next smallest square number with 5 as a factor is $(2 \times 5)^2 = 100$.

This is possible, one way of doing it is shown on the next page.



Key
B = Big
S = Small

10 by 10 Ans

B6. The letters a, b, c, d, e and f represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \qquad b + c = e \qquad \text{and} \qquad d + e = f.$$

Find all possible solutions for the values of a, b, c, d, e and f.

Solution

B6) Since a, b, c, d, e and f all represent different single digits, f must be less than 10. Also, since all the digits have to be different, the minimum value for d or e must be 3. Since the maximum value for f is 9, the maximum value for d or e must be 6 (6+3=9).

Since d and e must be different and larger than 3, the minimum value for f must be 7.

Now, with these details, we can begin to look at all the different combinations for a, b, c, d, e and f.

	$a+b=d$	$b+c=e$	$d+e=f$
$f=9$	$1+2=3$	$2+4=6$	$3+6=9$
	$4+2=6$	$2+1=3$	$6+3=9$
	$2+1=3$	$1+5=6$	$3+6=9$
	$5+1=6$	$1+2=3$	$6+3=9$
	$1+3=4$	$3+2=5$	$4+5=9$
	$2+3=5$	$3+1=4$	$5+4=9$

Since the only valid sums for making 9 are 3+6 and 4+5, these are all the possible solutions when f = 9

PTO

B6 solution (continued)

Now, let's try the next value for f, 8

$a+b=d$ $b+c=e$ $d+e=f$

$f=8$:

$2+1=3$ $1+4=5$ $3+5=8$

$4+1=5$ $1+2=3$ $5+3=8$

Since all the values must be different, these are the only 2 ways to make $f=8$. $4+4$ would be invalid because d and e must be different and using $1+2=3$ $2+3=5$ would also be invalid because d cannot equal c.

Finally, there are no possible ways for f to equal 7.

$2+1=3$ $1+3=4$ $3+4=7$ is invalid because d should not equal c and $1+2=3$ $2+2=4$ $3+4=7$ is also invalid for a similar reason. So, all the possible solutions are shown in the table below

a	b	c	d	e	f
1	2	4	3	6	9
4	2	1	6	3	9
2	1	5	3	6	9
5	1	2	6	3	9
1	3	2	4	5	9
2	3	1	5	4	9
2	1	4	3	5	8
4	1	2	5	3	8